

# **Wigner's Theorem**

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**ABSTRACT** A proof based on *The quantum theory of fields, vol. 1: Foundations* by Weinberg, S., & Greenberg, O. W. (1995).

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**QM Assumptions** (1) Physical states are vecs/rays  $\in \text{Hilbert sp } \mathcal{H}$ —some cmplx vec sp wi' a norm s.t.  $\xi\phi + \eta\psi \in \mathcal{H}, \forall \xi, \eta \in \mathbb{C}$   $(\phi, \psi) \in \mathbb{C}$

(i)  $(\Phi, \Psi) = (\Psi, \Phi)^*$  (ii)  $(\xi_1\Phi_1 + \xi_2\Phi_2, \eta\Psi) = \eta[\xi_1^*(\Phi_1, \Psi) + \xi_2^*(\Phi_2, \Psi)]$  (iii)  $(\Psi \neq 0, \Psi) > (0, 0) = 0$ .  
(2) Observables are Hermitian ops s.t.  $A(\xi\phi + \eta\psi) = \xi A\phi + \eta A\psi$ . (3) A ray  $\mathcal{R} \mid_{\exists \Psi}$  has a definite E-value  $a$   $A^\dagger = A$   $A\Psi = a\Psi$  an automorphism  $\text{Aut}(\mathcal{H})$

$\forall$  op  $A$ .<sup>\*</sup> Testing a syst in  $\mathcal{R}$  brings itself into one of the orthogonal rays  $\mathcal{R}_{n=1,2,\dots} \mid_{\exists \Psi_n}$  wi' transition prob  $P(\mathcal{R} \rightarrow \mathcal{R}_n) := |(\Psi, \Psi_n)|^2 = P(\mathcal{R}_n \rightarrow \mathcal{R})$ , s.t.  $\sum_n P(\mathcal{R} \rightarrow \mathcal{R}_n) = 1$  for a complete set  $\{\Psi_n\}$ .

**NB** (i) Pick a normalised vec  $\Psi$  as (a representative of) a ray  $\in \mathcal{H}$ . (ii) Define the **adjoint**  $A^\dagger$  via  $(\Phi, A^\dagger\Psi) := (A\Phi, \Psi)$  for a linear op  $A$  or  $(\Phi, A^\dagger\Psi) := \overline{(A\Phi, \Psi)^*}$  for an antilinear op  $A$ .

**Wigner's Theorem** An invertible transition-prob-preserving ray-transformation  $T$  s.t.

$$(0.1) \quad P\left(\underbrace{\mathcal{R}' \mid_{\exists \Psi' = U\Psi}}_{|(\Psi', \Psi'_n)|^2} = TR \leftrightarrow \mathcal{R}'_n \mid_{\exists \Psi'_n = U\Psi_n} = TR_n\right) = \underbrace{P(\mathcal{R} \leftrightarrow \mathcal{R}_n)}_{|(\Psi, \Psi_n)|^2}$$

$\Rightarrow$  a(n) (anti)unitary & (anti)linear op  $U$  on  $\mathcal{H}$  s.t.

$$(0.2) \quad \begin{aligned} & \underbrace{(U\Phi, U\Psi) = (\Phi, \Psi)}_{\Leftrightarrow U^\dagger U = 1} \quad \& \quad U(\xi\Phi + \eta\Psi) = \xi U\Phi + \eta U\Psi \\ & \text{or} \quad \underbrace{(U\Phi, U\Psi) = (\Phi, \Psi)^*}_{\Leftrightarrow U^\dagger U = 1} \quad \& \quad U(\xi\Phi + \eta\Psi) = \xi^* U\Phi + \eta^* U\Psi. \end{aligned}$$

*Proof.* A complete orthogonal set  $\{\Psi_n\}$   $\xrightarrow{|(\Psi'_n, \Psi'_n)|^2 \stackrel{(0.1)}{=} \delta_{n_1, n_2} \stackrel{(\Psi'_1, \Psi'_2) \geq 0}{\longrightarrow} (\Psi'_1, \Psi'_2) = \delta_{n_1, n_2}}$  another com-

plete orthogonal set  $\{U\Psi_n\}$ <sup>†</sup>  $\Rightarrow$  the expansions

$$(0.3) \quad \Psi \mid_{\in \mathcal{R}} = \sum_{n=1}^N C_n \Psi_n \quad \& \quad (U\Psi) \mid_{\in T\mathcal{R}} = \sum_{n=1}^N C'_n U\Psi_n$$

for a given  $\Psi$  & its counterpart  $U\Psi$ . To carry on, we must decide the *relative phases* for the transformed

\*An order- $N$  Hermitian matrix has  $N$  orthogonal E-vecs wi' distinct *real* E-values.

<sup>†</sup>Given a non-zero  $\Psi' \notin \{\Psi'_n\}$  but  $\perp$  all  $\Psi'_n$ , the 1-1 inverse map will take it back to some non-zero  $\Psi'' \notin \{\Psi_n\}$  s.t.  $|(\Psi_n, \Psi'')|^2 = |(\Psi'_n, \Psi')|^2 = 0 \Rightarrow$  impossible, as  $\{\Psi_n\}$  is already complete.

$$\text{basis } \{U\Psi_n\}: \left| \left( \sum_{i=1}^{l=1, \dots, N} \Psi_{n_i}, \Psi \right) \right|^2 \xrightarrow{(0.1)} \left| \left( U \sum_{i=1}^{l=1, \dots, N} \Psi_{n_i}, U\Psi \right) \right|^2 \xrightarrow[(0.3)]{\Psi = \Psi_{n_i=1, \dots, l}}$$

$$(0.4) \quad U \sum_{i=1}^{l=1, \dots, N} \Psi_{n_i} = \sum_{i=1}^{l=1, \dots, N} e^{i\theta_i} U\Psi_{n_i},$$

& our convention is  $\theta_{i=1, \dots, l} \equiv 0 \forall i \in \mathbb{N}$ . Now we can investigate the relation betw the 2 sets of expansion coefficients,  $\{C_n\}$  &  $\{C'_n\}$ . 1st,  $|\langle \Psi_n, \Psi \rangle|^2 \xrightarrow{(0.1)} |\langle U\Psi_n, U\Psi \rangle|^2 \xrightarrow{(0.3)}$

$$(0.5) \quad \left| \frac{C'_n}{C_n} \right| = 1.$$

$$\text{Then } \left| \left( \sum_{i=1}^2 \Psi_{n_i}, \Psi \right) \right|^2 \xrightarrow{(0.1)} \left| \left( U \sum_{i=1}^2 \Psi_{n_i} \xrightarrow{\text{phase convention}} \sum_{i=1}^2 U\Psi_{n_i}, U\Psi \right) \right|^2 \xrightarrow{(0.3)}$$

$$\begin{aligned} \left| \frac{\sum_{i=1}^2 C'_{n_i}}{\sum_{i=1}^2 C_{n_i}} \right| &= 1 \xrightarrow{(0.5)} \frac{|1 + C'_{n_2}/C'_{n_1}|^2 - 1 - |C'_{n_2}/C'_{n_1}|^2}{|1 + C_{n_2}/C_{n_1}|^2 - 1 - |C_{n_2}/C_{n_1}|^2} = 1 \\ &\quad \underbrace{\text{Re}(C'_{n_2}/C'_{n_1})/\text{Re}(C_{n_2}/C_{n_1})}_{\text{Re}(C'_{n_2}/C'_{n_1}) = |C_{n_2}/C_{n_1}|} \\ (0.6) \quad \xrightarrow[(0.5)]{|C'_{n_2}/C'_{n_1}| = |C_{n_2}/C_{n_1}|} \quad \frac{C_{n_2}}{C_{n_1}} &= \frac{C'_{n_2}}{C'_{n_1}} \text{ or } \left( \frac{C'_{n_2}}{C'_{n_1}} \right)^*. \end{aligned}$$

Next, think of  $\left| \left( \sum_{i=1}^{N \geq 3} \Psi_{n_i}, \Psi \right) \right|^2 \xrightarrow{(0.1)} \left| \left( U \sum_{i=1}^{N \geq 3} \Psi_{n_i} \xrightarrow{\text{phase convention}} \sum_{i=1}^{N \geq 3} U\Psi_{n_i}, U\Psi \right) \right|^2$ , wi'  $\{C_n\}$  &  $\{C'_n\}$  satisfy'g eq (0.6) in the way that  $C_{n_2}/C_{n_1} \equiv C'_{n_2}/C'_{n_1}$  or  $\equiv (C'_{n_2}/C'_{n_1})^*$ . To prove this ' $\equiv$ ', let's single out some  $C_{n_1}$ , & assume that  $C_{n_i}/C_{n_1} = C'_{n_i}/C'_{n_1} \forall i \in \{2, \dots, M < N\}$  &  $C_{n_i}/C_{n_1} = (C'_{n_i}/C'_{n_1})^* \forall i \in \{M+1, \dots, N\}$ . Then, from eqs (0.5) & (0.6),

$$\begin{aligned} 0 &= \left| 1 + \sum_{i=2}^N \frac{C'_{n_i}}{C'_{n_1}} \right|^2 - \left| 1 + \sum_{i=2}^N \frac{C_{n_i}}{C_{n_1}} \right|^2 = \sum_{i,j=2}^N \left( \frac{C'_{n_i} C'_{n_j}^* - C_{n_i} C_{n_j}^*}{|C_{n_1}|^2} + i \leftrightarrow j \right) \\ (0.7) \quad &= \sum_{i=2}^M \sum_{j=M+1}^N \left[ -4 \left( \text{Im} \frac{C_{n_i}}{C_{n_1}} \right) \text{Im} \frac{C_{n_j}}{C_{n_1}} \right] = (-4)^{(M-1)(N-M)} \underbrace{\left( \sum_{i=2}^M \text{Im} \frac{C_{n_i}}{C_{n_1}} \right)}_x \underbrace{\left( \sum_{j=M+1}^N \text{Im} \frac{C_{n_j}}{C_{n_1}} \right)}_y \end{aligned}$$

$\Rightarrow$  either  $x$  or  $y$  must  $\in \mathbb{R}$ , which is an unreasonable stronger statement (transcendental coefficient-constraint). In brief,  $\forall$  sym transformation  $T$ , the correspond'g  $U$  satisfies

$$(0.8) \quad U \sum_n C_n \Psi_n = \sum_n C_n U\Psi_n \quad \text{or} \quad \sum_n C_n^* U\Psi_n.$$

Our final step is to prove that the '=' in eq (0.8) is actually ' $\equiv$ ', which leads directly to the property (0.2). In fact, the mixed unitary-antiunitary case wi' a stronger constraint

$$(0.9) \quad U \underbrace{\sum_n A_n \Psi_n}_{\Psi} = \sum_n A_n U \Psi_n \quad \& \quad U \underbrace{\sum_n B_n \Psi_n}_{\Phi} = \sum_n B_n^* U \Psi_n$$

$$\xrightarrow[\text{(0.1)}]{|(\Psi, \Phi)|^2 = |(\Psi', \Phi')|^2} \underbrace{\sum_{n_1, n_2} (\text{Im } A_{n_1}^* A_{n_2}) \text{Im } B_{n_1}^* B_{n_2}}_{(|\sum_n A_n B_n^*|^2 - |\sum_n A_n B_n|^2) / \prod_{n_1, n_2} (-4)} = 0$$

is again unreasonable, as the probability preservation (0.1) is automatically satisfied wi' the overall (anti)unitary condition ' $\equiv$ '.  $\square$