Elements of Measures

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Abstract

Self-study notes on Axler's *Measure, Integration & Real Analysis*.[i] I share a succinct digest complemented by a bit of my own (naïve) comprehension (in some details for the *measure* part), with the hope of providing a beginner's perspective to fellow learners. Please refer to the original text for much greater interpretations.

Contents

[i]Free copy online! How generous. Thank you, professor Axler.

1 Measures

1.1 Outer measure on $\mathbb{R}$

1.1.1 Definition & good properties of outer measure

1.1 The ***outer measure***[ii][iii]

$$\mathring{\mu}_A := \inf\left\{ \sum_{k=1}^{\infty} \ell_{I_k} \;\middle|\; \{I_k\}_{k=1}^{\infty} \text{ is an open cover of } A \right\}$$

of $A \subseteq \overline{\mathbb{R}}$, with

$$\ell_I := \begin{cases} b-a & \text{if } \exists a \;\&\; b \in \mathbb{R} : a < b \wedge I = (a,b) \\ 0 & \text{if } I = \varnothing \\ \infty & \text{if } \exists a \in [-\infty, \infty) : I = \pm(a, \infty) \end{cases}$$

the ***length*** of an interval $I \subseteq \mathbb{R}$

Properties (outer measure's) 1. $\mathring{\mu}_{\forall \text{countable } C \subseteq \mathbb{R}} = 0$

2. $\mathring{\mu}_{\forall A \subseteq B \subseteq \mathbb{R}} \leq \mathring{\mu}_B$

3. $\mathring{\mu}_{t+A} = \mathring{\mu}_A$ $\forall$***translation*** $(t+A)$ of $A \subseteq \mathbb{R}$ by $t \in \mathbb{R}$

4. $\mathring{\mu}_{\bigcup_{k=1}^{\infty} A_k} \leq \sum_{k=1}^{\infty} \mathring{\mu}_{A_k}$ $\forall \{A_k \subseteq \mathbb{R}\}_{k=1}^{\infty}$

Proof. 1. $\forall \epsilon > 0$, an open cover $\left\{ I_k = c_k + {}^{(-\epsilon,\epsilon)}\!/_{2^k} \right\}_{k=1}^{\infty}$ of $C = \{c_k\}_{k=1}^{\infty}$

$\Rightarrow \mathring{\mu}_C \leq \sum_{k=1}^{\infty} \left(\ell_{I_k} = {}^{\epsilon}\!/_{2^{k-1}} \right) = 2\epsilon \xlongequal{\epsilon\text{'s arbitrariness}} 0.$

2. B's every cover covers A.

3. ℓ_I is translational invariant (by any distance t) $\forall$interval I.

4. $\forall \epsilon > 0$, pick an open cover $\left\{I_{j,k}\right\}_{j=1}^{\infty}$ $\forall A_{k \in \mathbb{Z}_{>0}} : \sum_{j=1}^{\infty} \ell_{I_{j,k}} - \mathring{\mu}_{A_k} \in \left[0, {}^{\epsilon}\!/_{2^k}\right]$. Then $\mathring{\mu}_{\bigcup_{k=1}^{\infty} A_k \subseteq \bigcup_{i=2}^{\infty} \left\{ I_i \equiv \bigcup_{(k,j) \in (\mathbb{Z}_{>0})^2; k+j=i} \left\{I_{j,k}\right\} \right\} = \bigcup_{k=1}^{\infty} \bigcup_{j=1}^{\infty} \left\{I_{j,k}\right\}} \leq \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \ell_{I_{j,k}} \leq \sum_{k=1}^{\infty} \mathring{\mu}_{A_k} \underset{0}{\underline{+\epsilon}}$ □

Remark. • $\mathbb{Q} \subseteq \mathbb{R}$ is countable $\Rightarrow \mathring{\mu}_{\mathbb{Q}} = 0$

• $\mathring{\mu}_{\varnothing} \xlongequal[\forall S \subseteq \mathbb{R},\ \mu_S \geq 0 \wedge \varnothing \leq S]{\text{properties 1–2}} 0.$

1.1.2 Outer measure of compact interval

1.1 $\mathring{\mu}_{[a,b]} = b - a$ $\forall a \;\&\; b \in \mathbb{R} : a < b$

Proof. 1. $\forall \epsilon > 0$, $\mathring{\mu}_{[a,b] \subseteq (a-\epsilon, b+\epsilon) \cup \varnothing \cup \varnothing \cup \cdots = (a-\epsilon, b+\epsilon)} \leq \mathring{\mu}_{(a-\epsilon, b+\epsilon)} = b - a \underset{0}{\underline{+2\epsilon}}$

2. (a) By Heine-Borel's theorem, every open cover $\{I_k\}_{k=1}^{\infty}$ of a closed bounded $[a,b] \subseteq \mathbb{R}$ has a finite subcover $\{I_k\}_{k=1}^{n}$ (b) Prove $\sum_{k=1}^{n} \ell_{I_k} \geq b - a$ by induction on $n \in \mathbb{Z}_{>0}$. Then $\sum_{k=1}^{\infty} \ell_{I_k} \geq \sum_{k=1}^{n} \ell_{I_k} \geq b - a \Rightarrow \mathring{\mu}_{[a,b]} \geq b - a$ □

Remark. $\mathring{\mu}_{(a,b) \subseteq \mathbb{R}} = \mathring{\mu}_{(a,b]} = \mathring{\mu}_{[a,b]} = \mathring{\mu}_{[a,b)}$.

1.2 *Every nontrivial (i.e.* $\exists a \;\&\; b \in I : a < b$*) interval* $I \subseteq \mathbb{R}$ *is uncountable*[iv]

[ii] $\sum_{k=1}^{\infty} t_k := \sum_{k=1}^{n \to \infty} t_k$ $\forall$sequence $\{t_k\}_{k=1}^{\infty} \equiv t_{k=1,2,\ldots}$

[iii] $\overline{\mathbb{R}} \equiv [-\infty, \infty] = \mathbb{R} \uplus \{\pm\infty\}$, with $\mathbb{R} \equiv (-\infty, \infty)$, and $\uplus$ a disjoint union

[iv] $\mathring{\mu}_{I \supseteq [a,b]} \geq \mathring{\mu}_{[a,b]} = b - a > 0$

1.1.3 Nonadditivity of outer meansure

1.3 $\exists A \,\&\, B \subseteq \mathbb{R} : \mathring{\mu}_{A \uplus B} \neq \mathring{\mu}_A + \mathring{\mu}_B$ ■

Proof. Partition $[-1,1]$ into equivalence classes $[a] := \{b \in [-1,1] \mid a - b \in \mathbb{Q}\}$, and pick $V \subseteq [-1,1] : |V \cap [a]| = 1 \; \forall a \in [-1,1]$.[v] Then $\{q_k\}_{k=1}^{\infty} \equiv [-2,2] \cap \mathbb{Q}$
$\Rightarrow [-1,1] \subseteq \biguplus_{k=1}^{\infty}(q_k + V) \subseteq [-3,3]$
$\Rightarrow \underbrace{\mathring{\mu}_{[-1,1]}}_{=2>0} \leq \mathring{\mu}_{\biguplus_{k=1}^{\infty}(q_k+V)} \xlongequal[\text{if } \mathring{\mu}_{A\uplus B}=\mathring{\mu}_A+\mathring{\mu}_B \; \forall A,B\subseteq\mathbb{R}]{\text{mathematical induction}} \sum_{k=1}^{\infty} \underbrace{\mathring{\mu}_{q_k+V}}_{\equiv \mathring{\mu}_V} = \underbrace{\left|\{q_k\}_{k=1}^{\infty}\right|}_{\infty} \cdot \mathring{\mu}_V \leq \underbrace{\mathring{\mu}_{[-3,3]}}_{=6<\infty}$
$\Rightarrow$ contradiction: $0 < \infty \cdot (\mathring{\mu}_V = 0) = 0$ □

1.2 Measurable spaces & maps

1.2.1 Motivation & definition of σ-algebra

1.4 $\underbrace{2^{\mathbb{R}} := \{S\}_{S\subseteq\mathbb{R}}}_{\mathbb{R}\text{'s power set}} \xrightarrow{\nexists \mu} \overline{\mathbb{R}}_{\geq 0}$:

1. $\mu_I = \ell_I \; \forall$ *open interval* $I \subseteq \mathbb{R}$
2. $\mu_{\biguplus_{k=1}^{\infty} A_k} = \sum_{k=1}^{\infty} \mu_{A_k} \; \forall \{A_k \subseteq \mathbb{R}\}_{k=1}^{\infty}$
3. $\mu_{t+A} = \mu_A \; \forall A \subseteq \mathbb{R} \; \forall t \in \mathbb{R}$ ■

Proof. μ has all $\mathring{\mu}$'s properties used to prove theorem 1.3 □

1.2 $\mathcal{S} \subseteq 2^X$ is a ***σ-algebra*** on a set X if

1. $X \backslash E \in \mathcal{S} \forall E \in \mathcal{S}$
2. $\varnothing \in \mathcal{S} \; (\Longleftrightarrow X = X\backslash\varnothing \in \mathcal{S})$
3. $\forall \{E_k \in \mathcal{S}\}_{k=1}^{\infty}, \bigcup_{k=1}^{\infty} E_k \in \mathcal{S} \; (\xLeftrightarrow{\text{de Morgan's laws}} \bigcap_{k=1}^{\infty} E_k = X \backslash \bigcup_{k=1}^{\infty} (X\backslash E_k) \in \mathcal{S})$.

$(X, \mathcal{S})$ is then called a ***measurable space***, and $E \in \mathcal{S}$ ***measurable sets*** ●
E.g. $\{\varnothing, X\}$ and 2^X are σ-algebras on X.
1.5 $\bigcap_{\mathcal{S}\in\{\mathcal{S}'\subseteq 2^X \mid \mathcal{S}' \text{ is a } \sigma\text{-algebra on } X \text{ containing } \mathcal{A}\}} \mathcal{S}$ *is the smallest σ-algebra on X containing* $\mathcal{A} \subseteq 2^X$ ■
Examples (of smallest σ-algebras)

1. $\{E \in X \mid E \text{ countable} \vee X\backslash E \text{ countable}\}$ on X containing $\{\{x\}\}_{x\in X}$.
2. $\{\varnothing, \mathbb{R}, (0,1), \mathbb{R}_{>0}, \mathbb{R}_{\leq 0} \uplus \mathbb{R}_{\geq 1}, \mathbb{R}_{\leq 0}, \mathbb{R}_{\geq 1}, \mathbb{R}_{<1}\}$ on $\mathbb{R}$ containing $\{(0,1), \mathbb{R}_{>0}\}$.

1.2.2 BOREL's subsets of $\mathbb{R}$

1.3 The set $\mathcal{B}$ of **Borel's** $B \subseteq \mathbb{R}$ is the smallest σ-algebra on $\mathbb{R}$ containing all open $G \subseteq \mathbb{R}$ ●
Examples (of $B \in \mathcal{B}$) • Every closed set, every countable $\{r_k \in \mathbb{R}\}_{k=1}^{\infty}$, and every half-open interval

- $\left\{ r \in \mathbb{R} \;\middle|\; \mathbb{R} \xrightarrow{f} \mathbb{R} \text{ is continuous at } r \right\}$ as an open-set intersection is 'BOREL'.

[v] $|V|$ denotes the order of a set V

1.2.3 Inverse images of measurable maps are measurable

1.4 $X \xrightarrow{f} \mathbb{R}$ is ***measurable*** on a measurable space $(X, \mathcal{S})$ if $f^{-1}_{\forall B \in \mathcal{B}} \in \mathcal{S}$[vi] ●

E.g. • The only measurable $X \xrightarrow{f} \mathbb{R}$ on the measurable space $(X, \{\varnothing, X\})$ are constant maps.

- Every $X \xrightarrow{f} \mathbb{R}$ is measurable on the measurable space $(X, 2^X)$.

- $\mathbb{R} \xrightarrow{f} \mathbb{R}$ is measurable on the measurable space $(\mathbb{R}, \{\varnothing, \mathbb{R}, \mathbb{R}_{<0}, \mathbb{R}_{\geq 0}\})$ iff f is constant respectively on $\mathbb{R}_{<0}$ and on $\mathbb{R}_{\geq 0}$.

- A ***characteristic map*** $X \xrightarrow{\chi_E} \mathbb{R}$ of $E \subseteq X$ with $\chi_{E;\forall x \in X} := \begin{Bmatrix} 1 & \text{if } x \in E \\ 0 & \text{if } x \notin E \end{Bmatrix}$ is measurable on a measurable space $(X, \mathcal{S})$ iff $E \in \mathcal{S} \Leftarrow \chi^{-1}_{E;B \subseteq \mathbb{R}} = \begin{cases} E & \text{if } 0 \notin B \ni 1 \\ X \backslash E & \text{if } 0 \in B \not\ni 1 \\ X & \text{if } 0 \in B \ni 1 \\ \varnothing & \text{if } 0 \notin B \not\ni 1. \end{cases}$ [vii]

1.6 *$X \xrightarrow{f} \mathbb{R}$ is measurable on a measurable space $(X, \mathcal{S}) \Leftarrow f^{-1}_{(\forall a \in \mathbb{R}, \infty)} \in \mathcal{S}$* ■

Proof. $\{A \subseteq \mathbb{R} \mid f^{-1}_A \in \mathcal{S}\}$ is a σ-algebra containing $\mathcal{B}$ □

Remark. The collection $\{\mathbb{R}_{>a}\}_{a \in \mathbb{R}}$ in the condition can be replaced by any $\mathcal{A} \subseteq 2^{\mathbb{R}}$: $\mathcal{B} \subseteq$ the smallest σ-algebra containing $\mathcal{A}$. *E.g.* $\mathcal{A} = \{(p, q]\}_{p,q \in \mathbb{Q}} \vee = \{(q, z]\}_{q \in \mathbb{Q}, z \in \mathbb{Z}} \vee = \{(q, q+1)\}_{q \in \mathbb{Q}} \vee = \{\mathbb{R}_{\geq q}\}_{q \in \mathbb{Q}}$ *etc.*

1.7 $\{E' \in \mathcal{S}\}_{E' \subseteq X'} = \{E \cap X'\}_{E \in \mathcal{S}}$ *is a σ-algebra on $X' \in \mathcal{S}$ $\forall \sigma$-algebra $\mathcal{S} \subseteq 2^X$* ■

1.5 $\forall X \subseteq \mathbb{R}, X \xrightarrow{f} \mathbb{R}$ is **Borel-*measurable*** if $f^{-1}_{\forall B \in \mathcal{B}} \in \mathcal{B}$ ●

1.8 *Every continuous $B \xrightarrow{f} \mathbb{R}$ is $\mathcal{B}$-measurable $\forall B \in \mathcal{B}$* ■

Proof. $f^{-1}_{(\forall a \in \mathbb{R}, \infty)} = \left(\bigcup_{b \in f^{-1}_{\mathbb{R}_{>a}}} (b - \delta_b, b + \delta_b)\right) \cap B \in \mathcal{B}$
$\Leftarrow f_{\forall b \in B} > a \; \exists \delta_b > 0 : f_{\forall x \in (b - \delta_b, b+\delta_b) \cap B} > a$ □

1.9 *Every increasing $B \xrightarrow{f} \mathbb{R}$ is $\mathcal{B}$-measurable $\forall B \in \mathcal{B}$* ■

Proof. $f^{-1}_{(\forall a \in \mathbb{R}, \infty)} \overset{b = \inf f^{-1}_{\mathbb{R}_{>a}}}{=\!=\!=} \mathbb{R}_{>b} \cap B \in \mathcal{B}$ □

1.10 *$X \xrightarrow{g \circ f} \mathbb{R}$ is measurable on a measurable space $(X, \mathcal{S})$ $\forall \mathcal{S}$-measurable $X \xrightarrow{f} \mathbb{R}$ $\forall \mathcal{B}$-measurable $Y \xrightarrow{g} \mathbb{R} : Y \supseteq f_X$* ■

E.g. $X \xrightarrow{f} \mathbb{R}$ is measurable on a measurable space $(X, \mathcal{S}) \Rightarrow$ so are $-f$, $f/_2$, $|f|$, f^2 *etc.*

[vi]$\forall X \xrightarrow{f} Y$, the ***inverse image*** $f^{-1}_A := \{x \in X \mid f_x \in A\} = X \backslash f^{-1}_{Y \backslash A}$ of $A \subseteq Y$. Besides, $f^{-1}_{\bigcirc_{A \in \mathcal{A}} A} \overset{\bigcirc = \bigcup, \bigcap}{=\!=\!=} \bigcirc_{A \in \mathcal{A}} f^{-1}_A \; \forall \mathcal{A} \subseteq 2^Y$, $(g \circ f)^{-1}_{\forall A \subseteq Z} = f^{-1}_{g^{-1}_A} \; \forall Y \xrightarrow{g} Z$

[vii]$\forall$measurable space $(X, \mathcal{S})$ $\forall x \in X$, Dirac's measure (*cf.* definition 1.7) $\mathcal{S} \xrightarrow{\delta_x : E \mapsto \chi_{E;x}} \overline{\mathbb{R}}_{\geq 0}$

1.11 $X \xrightarrow{f \,\&\, g} \mathbb{R}$ *are measurable on a measurable space* $(X, \mathcal{S}) \Rightarrow$ *so are* $f \pm g$, fg *and* $^f/_g$ $(g_{\forall x \in X} \neq 0$ *in the quotient)* ■

Proof. $fg = {}^{(f+g)^2 - f^2 - g^2}/_2$, $(f+g)^{-1}_{(\forall a \in \mathbb{R}, \infty)} = \bigcup_{q \in \mathbb{Q}} \left(f^{-1}_{\mathbb{R}_{>q}} \cap g^{-1}_{\mathbb{R}_{>a-q}} \right) \in \mathcal{S}$ □

1.12 $\exists f_{k \to \infty; \forall x \in X}$ *for a sequence* $\left\{ X \xrightarrow{f_k} \mathbb{R} \right\}_{k=1}^{\infty}$ *of measurable maps on a measurable space* $(X, \mathcal{S}) \Rightarrow \mathcal{S}$*-measurable* $X \xrightarrow{f: x \mapsto f_{k \to \infty; x}} \mathbb{R}$. ■

Proof. $f^{-1}_{(\forall a \in \mathbb{R}, \infty)} = \bigcup_{j=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcap_{k=m}^{\infty} f^{-1}_{k; \mathbb{R}_{>a+1/j}} \in \mathcal{S}$ □

1.6 $X \xrightarrow{f} \overline{\mathbb{R}}$ is ***measurable*** on a measurable space $(X, \mathcal{S})$ if $f^{-1}_{\forall B \in \overline{\mathcal{B}}} \in \mathcal{S}$,[viii] where $B \subseteq \overline{\mathbb{R}}$ is **Borel's** set if $B \cap \mathbb{R} \in \mathcal{B}$ (and $\overline{\mathcal{B}}$ is the collection of all such B) ●

1.13 *A sequence* $\left\{ X \xrightarrow{f_k} \overline{\mathbb{R}} \right\}_{k=1}^{\infty}$ *of measurable maps on a measurable space* $(X, \mathcal{S})$ $\Rightarrow \mathcal{S}$*-measurable* $X \xrightarrow{g \,\&\, h} \overline{\mathbb{R}} : g_{\forall x \in X} := \inf_{\{f_{k;x}\}_{k=1}^{\infty}}$, $h_{\forall x \in X} := \sup_{\{f_{k;x}\}_{k=1}^{\infty}}$ ■

Proof. $g_{\forall x \in X} = -\sup_{\{-f_{k;x}\}_{k=1}^{\infty}}$, $h^{-1}_{(\forall a \in \mathbb{R}, \infty]} = \bigcup_{k=1}^{\infty} f^{-1}_{k; \overline{\mathbb{R}}_{>a}} \in \mathcal{S}$ □

1.3 Measures & their properties

1.7 $\mathcal{S} \xrightarrow{\mu} \overline{\mathbb{R}}_{\geq 0}$ is a ***measure*** on a measurable space $(X, \mathcal{S})$ if $\mu_{\biguplus_{k=1}^{\infty} E_k} = \sum_{k=1}^{\infty} \mu_{E_k}$ $\forall \{E_k \in \mathcal{S}\}_{k=1}^{\infty}$. $(X, \mathcal{S}, \mu)$ is then called a ***measure space*** ●

Remark. $\mu_{E = E \uplus \varnothing \uplus \varnothing \uplus \cdots} = \mu_E + \sum_{k=2}^{\infty} \mu_{\varnothing} \Rightarrow \mu_{\varnothing} = 0$.

1.14 $\forall$*measure space* $(X, \mathcal{S}, \mu)$ $\forall \{E_k \in \mathcal{S}\}_{k=1}^{\infty}$

1. $E_1 \subseteq E_2 \Rightarrow \mu_{E_1} \leq \mu_{E_2} \wedge \mu_{E_2 \setminus E_1} \overset{\mu_{E_1} < \infty}{=\!=} \mu_{E_2} - \mu_{E_1}$
2. $\mu_{\bigcup_{k=1}^{\infty} E_k} \leq \sum_{k=1}^{\infty} \mu_{E_k}$
3. $E_{\forall k \in \mathbb{Z}_{>0}} \subseteq E_{k+1} \Rightarrow \mu_{\bigcup_{k=1}^{\infty} E_k} = \mu_{E_{k \to \infty}}$
4. $E_{\forall k \in \mathbb{Z}_{>0}} \supseteq E_{k+1} \wedge \mu_{E_1} < \infty \Rightarrow \mu_{\bigcap_{k=1}^{\infty} E_k} = \mu_{E_{k \to \infty}}$
5. $\mu_{E = E_1 \cap E_2} < \infty \Rightarrow \mu_{E_1 \cup E_2} = \mu_{E_1} + \mu_{E_2} - \mu_E$ ■

Proof. 1. (a) $\mu_{E_2 = E_1 \uplus (E_2 \setminus E_1)} = \mu_{E_1} + \mu_{E_2 \setminus E_1} \geq \mu_{E_1}$

(b) $\mu_{E_1} < \infty \Rightarrow \mu_{E_2} - \mu_{E_1} \geq \mu_{E_1} - \mu_{E_1} = 0$

2. $\mu_{\bigcup_{k=1}^{\infty} E_k = \biguplus_{k=1}^{\infty} (E_k \setminus D_k)} = \sum_{k=1}^{\infty} \left(\mu_{E_k \setminus D_k} \leq \mu_{E_k} \right)$ with $D_{\forall k \in \mathbb{Z}_{>0}} = \bigcup_{j=1}^{k-1} E_k \overset{k=1}{=\!=} \varnothing$

3. Say $\mu_{E_{\forall k \in \mathbb{Z}_{>0}}} < \infty$, as otherwise both sides of the equation are ∞. Let $E_0 = \varnothing$, $\mu_{\bigcup_{k=1}^{\infty} E_k = \biguplus_{j=1}^{\infty} (E_j \setminus E_{j-1})} = \left(\sum_{j=1}^{\infty} \equiv \sum_{j=1}^{k \to \infty} \right) \left(\mu_{E_j \setminus E_{j-1}} = \mu_{E_j} - \mu_{E_{j-1}} \right) = \mu_{E_{k \to \infty}}$

4. $\mu_{E_1} - \mu_{\bigcap_{k=1}^{\infty} E_k} = \mu_{E_1 \setminus \bigcap_{k=1}^{\infty} E_k = \bigcup_{k=1}^{\infty} (E_1 \setminus E_k)} \overset{\text{property 3}}{=\!=\!=} \mu_{E_1 \setminus E_{k \to \infty}} = \mu_{E_1} - \mu_{E_{k \to \infty}}$

5. $\mu_{E_1 \cup E_2 = [\biguplus_{k=1}^{2} (E_k \setminus E)] \uplus E} = \left[\sum_{k=1}^{2} \left(\mu_{E_k \setminus E} = \mu_{E_k} - \mu_E \right) \right] + \mu_E = \mu_{E_1} + \mu_{E_2} - \mu_E$ □

[viii] $X \xrightarrow{f} \overline{\mathbb{R}}$ is measurable on a measurable space $(X, \mathcal{S}) \Leftarrow f^{-1}_{(\forall a \in \overline{\mathbb{R}}, \infty]} \in \mathcal{S}$

1.4 Lebesgue's measure

1.15 $\mathring{\mu}_{A\uplus B} = \mathring{\mu}_{\forall A\subseteq\mathbb{R}} + \mathring{\mu}_{\forall B\in\mathscr{B}}$ ■

Proof. Need to show $\mathring{\mu}_{A\uplus B} \geq \mathring{\mu}_A + \mathring{\mu}_B$.

1. $\mathring{\mu}_{A\uplus B} = \mathring{\mu}_{\forall A\subseteq\mathbb{R}} + \mathring{\mu}_{\forall \text{open } B\subseteq\mathbb{R}}$ Say $\mathring{\mu}_B < \infty$.

(a) If B is an open interval $(a,b)\subseteq\mathbb{R}$, then $\forall$open cover

$$\underbrace{\left\{a + {}^{(-\epsilon,\epsilon)}/_4, b + {}^{(-\epsilon,\epsilon)}/_4\right\}}_{I_0} \cup \{\underbrace{I_k\cap\mathbb{R}_{<a}}_{J_k}\}_{k=1}^{\infty} \uplus \{\underbrace{I_k\cap(a,b)}_{K_k}\}_{k=1}^{\infty} \uplus \{\underbrace{I_k\cap\mathbb{R}_{>b}}_{L_k}\}_{k=1}^{\infty}$$

of $A\uplus B$, $\sum_{k=0}^{\infty}\ell_{I_k} \xlongequal[I_0\cup\{J_k,L_k\}_{k=1}^{\infty}\supseteq A]{\{K_k\}_{k=1}^{\infty}\supseteq B} \underbrace{\overset{\epsilon\to 0}{\overline{\ell_{I_0}}} + \sum_{k=1}^{\infty}\left(\ell_{J_k}+\ell_{L_k}\right)}_{\geq\mathring{\mu}_A} + \underbrace{\sum_{k=1}^{\infty}\ell_{K_k}}_{\geq\mathring{\mu}_B} \Rightarrow \mathring{\mu}_{A\uplus B}\geq\mathring{\mu}_A+\mathring{\mu}_B$.

(b) If $B = \biguplus_{k=1}^{\infty} I_k$ for some open sequence $\{I_k\subseteq\mathbb{R}\}_{k=1}^{\infty}$, then
$\mathring{\mu}_{A\uplus B} \geq \underbrace{\mathring{\mu}_{A\,\biguplus_{k=1}^{\forall z\in\mathbb{Z}_{>0}} I_k} = \mathring{\mu}_A + \sum_{i=1}^{z}\ell_{I_k}}_{\text{by property (a) and induction on z}} \Rightarrow \mathring{\mu}_{A\uplus B} \geq \mathring{\mu}_A + \left(\sum_{k=1}^{\infty}\ell_{I_k}\geq\mathring{\mu}_B\right)$.

2. $\mathring{\mu}_{A\uplus B} = \mathring{\mu}_{\forall A\subseteq\mathbb{R}} + \mathring{\mu}_{\forall \text{closed } B\subseteq\mathbb{R}}$ $\forall$open cover $\{I_k\subseteq\mathbb{R}\}_{k=1}^{\infty}$ of $A\uplus B$,
$\sum_{k=1}^{\infty}\ell_{I_k} \geq \mathring{\mu}_{G=\bigcup_{k=1}^{\infty}I_k=(G\setminus B)\uplus B} \xlongequal[G\setminus B=G\cap(\mathbb{R}\setminus B)\text{ is open}]{\text{step 1}} \mathring{\mu}_{G\setminus B\supseteq A} + \mathring{\mu}_B \geq \mathring{\mu}_A + \mathring{\mu}_B$
$\Rightarrow \mathring{\mu}_{A\uplus B}\geq\mathring{\mu}_A+\mathring{\mu}_B$.

3. $\mathscr{L} := \left\{L\subseteq\mathbb{R} \,\middle|\, \forall\epsilon>0\,\exists\text{ closed }F\subseteq L : \mathring{\mu}_{L\setminus F}<\epsilon\right\}$ is a σ-algebra containing $\mathbb{R}$'s all closed, and thus all open, all Borel's (and all 0-outer-measure) subsets Since $\mathscr{L}$ ($\ni\varnothing$, as $\varnothing$ is both open and closed) ***is closed under***

Countable intersection $L_0 = \bigcap_{k=1}^{\infty}L_k\in\mathscr{L}\;\forall\{L_k\in\mathscr{L}\}_{k=1}^{\infty} \Leftarrow \forall\epsilon>0$
$\exists$ closed $F_{\forall k\in\mathbb{Z}_{>0}}\subseteq L_k : \mathring{\mu}_{L_k\setminus F_k} < {}^{\epsilon}/_{2^k} \wedge \mathring{\mu}_{L_0\setminus(\text{closed }\bigcap_{k=1}^{\infty}F_k)=\bigcup_{k=1}^{\infty}(L_0\setminus F_k)\subseteq\bigcup_{k=1}^{\infty}(L_k\setminus F_k)} < \epsilon$.

Complementation $\forall L\in\mathscr{L}\;\forall\epsilon>0$

(a) If $\mathring{\mu}_L<\infty$, then $\exists$ closed $F\subseteq L\subseteq$ open $G : \epsilon$
$> \left({}^{\epsilon}/_2 > \mathring{\mu}_G - \mathring{\mu}_L\right) + \left({}^{\epsilon}/_2 > \mathring{\mu}_{L\setminus F} = \mathring{\mu}_L - \mathring{\mu}_F\right) = \mathring{\mu}_G - \mathring{\mu}_F$
$= \mathring{\mu}_{G\setminus F\supseteq G\setminus L=(\mathbb{R}\setminus L\supseteq\mathbb{R}\setminus G)\setminus(\mathbb{R}\setminus G)} \geq \mathring{\mu}_{(\mathbb{R}\setminus L)\setminus(\text{closed }\mathbb{R}\setminus G)}$.

(b) If $\mathring{\mu}_L=\infty$, $\mathring{\mu}_{L_{\forall k\in\mathbb{Z}_{>0}}=L\cap[-k,k]\in\mathscr{L}} < \infty$
$\xRightarrow{\text{step (a)}} \mathbb{R}\setminus L_{\forall k\in\mathbb{Z}_{>0}}\in\mathscr{L}\Rightarrow\mathbb{R}\setminus L = \bigcap_{k=1}^{\infty}\left(\mathbb{R}\setminus L_k\right)\in\mathscr{L}$.

4. $\forall\epsilon>0\,\exists$ closed $F\subseteq B : \mathring{\mu}_{B\setminus F}<\epsilon\wedge\mathring{\mu}_{A\uplus B}\geq\mathring{\mu}_{A\uplus F} = \mathring{\mu}_A + \left(\mathring{\mu}_F = \mathring{\mu}_B - \mathring{\mu}_{B\setminus F}\geq\mathring{\mu}_B\right)$ □

1.16 $\exists B\subseteq\mathbb{R} : \mathring{\mu}_B<\infty\wedge B$ *is not* Borel*'s set* ■

Proof. By theorems 1.3, 1.15 □

1.17 $(\mathbb{R},\mathscr{B},\mathring{\mu})$ *is a measure space* ■

Proof. $\forall\{B_k\in\mathscr{B}\}_{k=1}^{\infty}$, $\mathring{\mu}_{\biguplus_{k=1}^{\infty}B_k} \geq \underbrace{\mathring{\mu}_{\biguplus_{k=1}^{\forall z\in\mathbb{Z}_{>0}}B_k} = \sum_{k=1}^{z}\mathring{\mu}_{B_k}}_{\text{by theorem 1.15 and induction on z}} \Rightarrow \mathring{\mu}_{\biguplus_{k=1}^{\infty}B_k}\geq\sum_{k=1}^{\infty}\mathring{\mu}_{B_k}$ □

1.8 $A\subseteq\mathbb{R}$ is **Lebesgue-*measurable***

$\iff$ $\exists B^-\in\mathscr{B} : B^-\subseteq A\wedge\mathring{\mu}_{A\setminus B^-}=0$

$\iff$ $\forall\epsilon>0\,\exists$ closed $F\subseteq A : \mathring{\mu}_{A\setminus F}<\epsilon$

$\iff$ $\exists\{\text{closed }F_k\subseteq A\}_{k=1}^{\infty} : \mathring{\mu}_{A\setminus\bigcup_{k=1}^{\infty}F_k}=0$

$\iff \exists\{\text{open } G_k \supseteq A\}_{k=1}^{\infty} : \mathring{\mu}_{\bigcap_{k=1}^{\infty} G_k \setminus A} = 0$

$\iff \forall \epsilon > 0 \, \exists \text{ open } G \supseteq A : \mathring{\mu}_{G\setminus A} < \epsilon$

$\iff \exists B^+ \in \mathscr{B} : B^+ \supseteq A \wedge \mathring{\mu}_{B^+\setminus A} = 0$

$\iff \mathring{\mu}_{(-n,n)\cap A} + \mathring{\mu}_{(-n,n)\setminus A} = 2n \, \forall n \in \mathbb{Z}_{>0}$ ■

Proof. $\mathring{\mu}_{A\setminus B^-} = 0 = \mathring{\mu}_{B^+\setminus A} \xrightarrow[(A\setminus B^-)\uplus B^- = A = B^+ \cap [\mathbb{R}\setminus(B^+\setminus A)]]{B^\pm \in \mathscr{B} \subseteq \mathscr{L}} A\setminus B^- \;\&\; B^+\setminus A \;\&\; A \;\&\; \mathbb{R}\setminus A \in \mathscr{L}$

$\xrightarrow{\exists F \subseteq \bigcup_{k=1}^{\infty} F_k \subseteq A \subseteq \bigcap_{k=1}^{\infty} G_k \subseteq G} \mathring{\mu}_{A\setminus F} \;\&\; \mathring{\mu}_{G\setminus A = (\mathbb{R}\setminus A \supseteq \mathbb{R}\setminus G)\setminus(\mathbb{R}\setminus G)} < \epsilon \to 0^+$ *etc.* □

Remark. The σ-algebra $\mathscr{L}$ in theorem 1.15.3 is the collection of $\mathbb{R}$'s all $\mathscr{L}$-measurable subsets.

1.18 $(\mathbb{R}, \mathscr{L}, \mathring{\mu})$ *is a measure space (dubbed* Lebesgue's*)* ■

Proof. $\forall \{L_k \in \mathscr{L}\}_{k=1}^{\infty} \, \exists \left\{B_k \in \mathscr{B} \;\middle|\; L_k = B_k \uplus (L_k \setminus B_k)\right\}_{k=1}^{\infty} : \mathring{\mu}_{L_{\forall k \in \mathbb{Z}_{>0}} \setminus B_k} = 0$

$\wedge \; \mathring{\mu}_{\biguplus_{k=1}^{\infty} L_k} \geq \mathring{\mu}_{\biguplus_{k=1}^{\infty} B_k} \overset{\text{theorem 1.17}}{=\!=\!=} \sum_{k=1}^{\infty} (\mathring{\mu}_{B_k} \geq \mathring{\mu}_{L_k})$ □

Remark. $\forall A \subseteq \mathbb{R}$ with $\mathring{\mu}_A < \infty$, $A \in \mathscr{L} \iff \forall \epsilon > 0 \, \exists G = \biguplus_{k=1}^{n < \mathbb{Z}_{>0}} G_k$ with $G_{k=1,\ldots n}$ bounded open intervals: $\mathring{\mu}_{A\setminus G} + \mathring{\mu}_{G\setminus A} < \epsilon$. Practically, this means that every $B \in \mathscr{B}$ with $\mathring{\mu}_B < \infty$ is almost a finite disjoint union of bounded open intervals.

1.5 Convergence of measurable maps

1.5.1 Pointwise convergence is almost uniform convergence

1.9 $\left\{X \xrightarrow{f_k} \mathbb{R}\right\}_{k=1}^{\infty}$ ***converges*** to $X \xrightarrow{f} \mathbb{R}$

Pointwise (on** X**) if $f_{k\to\infty;\forall x \in X} = f_x$

Uniformly if $\forall \epsilon > 0 \, \exists n \in \mathbb{Z}_{>0} : \left|f_{\forall k \geq n; \forall x \in X} - f_x\right| < \epsilon$ ●

E.g. $\left\{[-1,1] \xrightarrow{f_k} \mathbb{R} \;\middle|\; f_{k;x} = \begin{cases} 1 - k|x| & \text{if } |x| \in [0, 1/k] \\ 0 & \text{if } |x| \in (1/k, 1] \end{cases}\right\}_{k=1}^{\infty}$ converges pointwise but not uniformly to $[-1,1] \xrightarrow{f : x \mapsto \delta_{0,x}} \mathbb{R}$.

1.19 $\left\{X \xrightarrow{f_k} \mathbb{R} \;\middle|\; f_{\forall j \in \mathbb{Z}_{>0}} \text{ continuous at } x \in X\right\}_{k=1}^{\infty}$ *converges uniformly to* $X \xrightarrow{f} \mathbb{R}$ $\Rightarrow f$ *continuous at* x ■

Proof. $\forall \epsilon > 0 \, \exists \delta > 0 : \left|f_{\forall x' \in (x-\delta, x+\delta)\cap X} - f_x\right| < \epsilon$, because

$\left|f_{x'} - f_x\right| \leq \left|f_{x'} - f_{j;x'}\right| + \left|f_{j;x'} - f_{j;x} < \epsilon'\right| + \left|f_{j;x} - f_x\right| \forall j \in \mathbb{Z}_{>0} \, \forall \epsilon' \in (0, \epsilon)$

$\xrightarrow{\left|f_{\exists n \in \mathbb{Z}_{>0}; \forall x'' \in X} - f_{x''}\right| < (\epsilon - \epsilon')/2} \left|f_{x'} - f_x\right| < \left|f_{x'} - f_{n';x'}\right| + \epsilon' + \left|f_{n;x} - f_x\right| < \epsilon$ □

Theorem (EGOROV'S) $\forall$*measure* $\mathcal{S} \xrightarrow{\mu} \mathbb{R}_{\geq 0}$ *on a measurable space* $(X, \mathcal{S})$ $\exists E \subseteq X$: $\mu_{X\setminus E} \in [0, \forall \epsilon > 0) \wedge \left\{\mathcal{S}\text{-measurable } X \xrightarrow{f_k} \mathbb{R}\right\}_{k=1}^{\infty}$ *converges to* $X \xrightarrow{f : x \mapsto f_{k\to\infty;x}} \mathbb{R}$ *uniformly on* E ■

Proof. $f_{k\to\infty;\forall x \in X} = f_x \xrightarrow{g_k \equiv f_k - f} \bigcup_{m=1}^{\infty} \left(A_{m;\forall n \in \mathbb{Z}_{>0}} := \bigcap_{k=m}^{\infty} g^{-1}_{k;(-1/n, 1/n)}\right) = X$, where

$A_{m;n} \in \mathcal{S}$ as $X \xrightarrow{g_{\forall k \in \mathbb{Z}}} \mathbb{R}$ is $\mathcal{S}$-measurable (by theorems 1.12, 1.11), and $\{A_{m,n}\}_{m=1}^{\infty}$ is an increasing sequence $\xRightarrow{\text{theorem 1.14.3}} \mu_X = \mu_{A_{m\to\infty;n}}$; *i.e.* $\mu_X - \mu_{\exists m_n \in \mathbb{Z}_{>0}} < {}^{\epsilon}/_{2^n}$. Thus $\mu_{X\setminus(E=\bigcap_{n=1}^{\infty} A_{m_n;n})=\bigcup_{n=1}^{\infty}(X\setminus A_{m_n;n})} \leq \sum_{n=1}^{\infty} \mu_{X\setminus A_{m_n;n}} < \epsilon$, and $\{f_k\}_{k=1}^{\infty}$ converges to f uniformly on $E \subseteq A_{m_n;\forall n \in \mathbb{Z}_{>0}}$, as $\forall \epsilon' > 0\, \exists n \in \mathbb{Z}_{>0} : \left|g_{\forall k \in \mathbb{Z}_{>0};\forall x \in E}\right| < {}^1/_n < \epsilon'$ □

1.5.2 Approximation by simple maps

1.10 A map is ***simple*** if it takes only finitely many values ●

E.g. A simple $X \xrightarrow{f=\sum_{k=1}^{n} c_k \chi_{E_k}} \mathbb{R}$ (measurable) on a measurable space $(X, \mathcal{S})$, with $c_{k=1,\dots,n}$ the distinct values $\in \mathbb{R}_{\neq 0}$ of f, and $E_{k=1,\dots,n} = f^{-1}_{\{c_k\}} \in \mathcal{S}$.

1.20 *$\forall$ measurable $X \xrightarrow{f} \overline{\mathbb{R}}$ on a measurable space $(X, \mathcal{S})$*

$\exists \left\{ \text{simple } \mathcal{S}\text{-measurable} X \xrightarrow{f_k} \mathbb{R} \;\middle|\; \left|f_{\forall j \in \mathbb{Z}_{>0};\forall x \in X}\right| \leq \left|f_{j+1;x}\right| \leq \left|f_x\right| \right\}_{k=1}^{\infty}$ *converging pointwise (uniformly for bounded* f*) to* f ■

E.g. $\left\{ f_{k;\forall x \in X} = \left(\left|f_{k;x}\right| = \begin{cases} {}^m/_{2^k} & \text{if } \exists m \in \mathbb{Z} : \left|f_x\right| \in [0,k] \cap {}^{[m,m+1)}/_{2^k} \\ k & \text{if } \left|f_x\right| \in (k, \infty) \end{cases} \right) \text{sign}_{f_x} \right\}_{k=1}^{\infty}$ is a desired sequence of simple $\mathcal{S}$-measurable ($f^{-1}_{[0,k]\cap{}^{[m,m+1)}/_{2^k}} \in \mathcal{S}$ *etc.* $\Leftarrow$ $\mathcal{S}$-measurable f) maps: $\left|f_{\forall k \in \mathbb{Z}_{>0};\forall x \in X} - f_x\right| \leq {}^1/_{2^k}$ if $\left|f_x\right| \in [0,k]$.

1.21 *$\forall$ continuous $F \xrightarrow{f} \mathbb{R}$ on a closed $F \subseteq \mathbb{R}\, \exists$ continuous $\mathbb{R} \xrightarrow{\bar{f}} \mathbb{R} : \bar{f}\big|_F = f$* ■

E.g. $\exists \{$open interval $I_k\}_{k=1}^{\infty} : \mathbb{R}\setminus F = \biguplus_{k=1}^{\infty} I_k$. $\bar{f}\big|_{I_k} := f_a \vee :=$ linear map connecting f_b & f_c for $I_{k \in \mathbb{Z}_{>0}} = \pm(a, \infty) \vee = (b, c)$.

1.5.3 Borel's measurability is almost continuity

Theorem (Lusin's) *$\mathcal{B}$-measurable* $E \xrightarrow{f} \mathbb{R} \Rightarrow \forall \epsilon > 0\, \exists$

- *closed* $F \subseteq \mathbb{R} : \mathring{\mu}_{E\setminus F} < \epsilon$
- *continuous* $\mathbb{R} \xrightarrow{\bar{f}} \mathbb{R} : \bar{f}\big|_F = f\big|_F$ ■

Proof. 1. Prove the theorem for $\left(E \xrightarrow{f} \mathbb{R}\right) = \left(\mathbb{R} \xrightarrow{\bar{f}} \mathbb{R}\right)$ 1st.

(a) Say $f = \sum_{k=1}^{n} c_k \chi_{B_k} \xlongequal[B_0=\mathbb{R}\setminus\bigcup_{k=1}^{n} B_k \in \mathcal{B}]{c_0=0} \sum_{k=0}^{n} c_k \chi_{B_k}$ of distinct $c_{k=1,\dots,n} \in \mathbb{R}_{\neq 0}$ and disjoint $B_{k=0,\dots,n} \in \mathcal{B}$. $\forall \epsilon > 0$, theorem 1.8 $\Rightarrow$ '$\forall k \in \{1, \dots, n\}\, \exists$ closed $F_k \subseteq B_k \subseteq$ open $G_k : \mathring{\mu}_{G_k\setminus B_k} < {}^{\epsilon}/_{2n} > \mathring{\mu}_{B_k\setminus F_k} \wedge \mathring{\mu}_{G_k\setminus F_k=(G_k\setminus B_k)\uplus(B_k\setminus F_k)} < {}^{\epsilon}/_n$'

$\Rightarrow$ closed $F \xlongequal{F_0=\mathbb{R}\setminus\bigcup_{k=1}^{n} G_k} \biguplus_{k=0}^{n} F_k : \mathring{\mu}_{\mathbb{R}\setminus F \subseteq \bigcup_{k=1}^{n}(G_k\setminus F_k)} < \epsilon$

$\wedge\ f\big|_F$ continuous (as $f\big|_{F_{\forall k \in \{0,\dots,n\}} \subseteq B_k} \equiv c_k$ is continuous)

(b) $\forall \mathcal{B}$-measurable $\mathbb{R} \xrightarrow{f} \mathbb{R}$

i. Theorem 1.20 $\Rightarrow \exists \left\{ \text{simple } \mathcal{B}\text{-measurable} \mathbb{R} \xrightarrow{f_k} \mathbb{R} \right\}_{k=1}^{\infty} : f_{k\to\infty;\forall x \in X} = f_x$. $\forall \epsilon > 0$,

step 1.(a) $\Rightarrow$ '$\forall k \in \mathbb{Z}_{>0}\ \exists$ closed $C_k \subseteq \mathbb{R} : \mathring{\mu}_{\mathbb{R}\setminus C_k} < \epsilon/2^{k+1} \wedge f_k|_{C_k}$ continuous'
$\Rightarrow f_{\forall k\in\mathbb{Z}_{>0}}|_{C=\bigcap_{j=1}^{\infty} C_j}$ continuous: $\mathring{\mu}_{(\mathbb{R}\setminus C)=\bigcup_{k=1}^{\infty}(\mathbb{R}\setminus C_k)} < \epsilon/2$.

ii. $\forall n \in \mathbb{Z}_{>0},\quad f_{k\to\infty;\forall x\in(n,n+1)} = f_x \xRightarrow{\text{Egorov's theorem}} \exists B_n \in \mathscr{B} : \mathring{\mu}_{(n,n+1)\setminus(B_n\subseteq(n,n+1))} < \epsilon/2^{|n|+3} \wedge \left\{f_k|_{B_n}\right\}_{k=1}^{\infty}$ converges to $f|_{B_n}$ uniformly on $C \cap B_n$.

iii. $f_{\forall k\in\mathbb{Z}_{>0}}|_{(C\cap B_{\forall n\in\mathbb{Z}_{>0}})\subseteq C\subseteq C_k}$ continuous $\xRightarrow{\text{theorem 1.19}} (f = f_{k\to\infty})|_{C\cap B_n}$ continuous
$\Rightarrow f|_{D=\bigcup_{n\in\mathbb{Z}_{>0}}(C\cap B_n)}$ continuous, where theorem 1.8
$\Rightarrow \mathring{\mu}_{D\setminus\exists \text{ closed } F\subseteq D\in\mathscr{L}} < \epsilon - \underbrace{\mathring{\mu}_{\mathbb{R}\setminus D=(\mathbb{R}\setminus C)\cup\left[\mathbb{R}\setminus\left(\bigcup_{n\in\mathbb{Z}_{>0}} B_n\right)\subseteq\mathbb{Z}_{>0}\cup\left(\bigcup_{n\in\mathbb{Z}_{>0}}(n,n+1)\setminus B_n\right)\right]}}_{>0}$

$\wedge\ \mathring{\mu}_{\mathbb{R}\setminus F=(\mathbb{R}\setminus D)\uplus(D\setminus F)} = \mathring{\mu}_{\mathbb{R}\setminus D} + \mathring{\mu}_{D\setminus F} < \epsilon \wedge f|_{F\subseteq D}$ continuous.

2. $\forall\epsilon > 0$, consider an extension $\mathbb{R} \xrightarrow{\tilde{f}:=\chi_E\cdot f} \mathbb{R}$ of $E \xrightarrow{f} \mathbb{R}$, then step 1
$\Rightarrow$ '$\exists$ closed $C \subseteq \mathbb{R} : \mathring{\mu}_{\mathbb{R}\setminus C} < \epsilon \wedge \tilde{f}|_C$ continuous'
$\Rightarrow$ '$\exists$ closed $F \subseteq C\cap E : \mathring{\mu}_{(C\cap E)\setminus F} < \underbrace{\epsilon - \mathring{\mu}_{\mathbb{R}\setminus C}}_{>0} \wedge \mathring{\mu}_{E\setminus F=[(C\cap E)\setminus F]\uplus[(E\setminus C)\subseteq(\mathbb{R}\setminus C)]} < \epsilon$

$\wedge\ \tilde{f}|_{F\subseteq E} = f|_F$ continuous'
$\xRightarrow{\text{theorem 1.21}} \exists$ continuous $\mathbb{R} \xrightarrow{\bar{f}} \mathbb{R} : \bar{f}|_F = f$ □

Remark. $\biguplus_{k=1,\dots,n} F_k \xrightarrow{f} \mathbb{R}$ with closed $F_{k=1,\dots,n} \subseteq \mathbb{R}$ and continuous $f|_{F_{k=1,\dots,n}}$ is continuous.

1.5.4 Lebesgue's measurability is almost Borel's measurability

1.11 $\forall X \subseteq \mathbb{R}, X \xrightarrow{f} \mathbb{R}$ is ***Lebesgue-measurable*** if $f^{-1}_{\forall B\in\mathscr{B}} \in \mathscr{L}$

1.22 $\forall\mathscr{L}$*-measurable* $\mathbb{R} \xrightarrow{f} \mathbb{R}\ \exists\ \mathscr{B}$*-measurable* $\mathbb{R} \xrightarrow{g} \mathbb{R} : \mathring{\mu}_{\{x\in\mathbb{R}\,|\,g_x\neq f_x\}} = 0$

Proof. $\mathscr{L}$-measurable $\mathbb{R} \xrightarrow{f} \mathbb{R} \xRightarrow{\text{theorem 1.20}} \exists\left\{\text{simple } \mathscr{L}\text{-measurable} \mathbb{R} \xrightarrow{f_k} \mathbb{R}\right\}_{k=1}^{\infty}$: $f_{k\to\infty;\forall x\in X} = f_x \wedge f_{\forall k\in\mathbb{Z}_{>0}} = \sum_{j=1}^{\exists n\in\mathbb{Z}_{>0}} c_j\chi_{A_j}$ of distinct $c_{j=1,\dots,n} \in \mathbb{R}_{\neq 0}$ and disjoint $A_{j=1,\dots,n} \in \mathscr{L}$. Theorem 1.8 $\Rightarrow$ '$\forall j \in \{1,\dots,n\}\ \exists B_j \in \mathscr{B} : \mathring{\mu}_{A_j\setminus(B_j\subseteq A_j)} = 0$'
$\Rightarrow \mathscr{B}$-measurable $g_{\forall k\in\mathbb{Z}_{>0}} = \sum_{j=1}^{n} c_j\chi_{B_j} : \mathring{\mu}_{\epsilon_k=\{x\in\mathbb{R}\,|\,g_{k;x}\neq f_{k;x}\}} = 0$. Thus $g_{k\to\infty;\forall x\in E} = f_x$ with $\mathring{\mu}_{\mathbb{R}\setminus\left(E=\{x\in\mathbb{R}\,|\,\exists g_{k\to\infty;x}\}\right)\subseteq\bigcup_{k=1}^{\infty}\epsilon_k} = 0 \Rightarrow \exists g_{\forall x\in\mathbb{R}} = \left(\chi_E\cdot g_{k\to\infty}\right)_x$
$\xRightarrow[\text{theorem 1.12}]{\mathscr{B}\text{-measurable }\left(\chi_E\cdot g_{\forall k\in\mathbb{Z}_{>0}}\right)} \mathscr{B}$-measurable $g : \mathring{\mu}_{\{x\in\mathbb{R}\,|\,g_x\neq f_x\}\subseteq\bigcup_{k=1}^{\infty}\epsilon_k} = 0$ □

2 Integration

2.1 Integration with respect to a measure

2.1.1 Integration of nonnegative maps

2.1 $\{A_j \in \mathcal{S} \mid \biguplus_{k=1}^m A_k = X\}_{j=1}^{m\in\mathbb{Z}_{>0}}$ is an ***S*-partition** of a measurable space $(X, \mathcal{S})$

2.2 The ***integral*** $\int f\,d\mu := \sup_{\mathcal{P}}\{\underbrace{\mathcal{L}_{f,\mathcal{P}=\{A_j\}_{j=1}^m} := \sum_{j=1}^m \mu_{A_j} \inf_{A_j} f}_{\text{Lebesgue's lower sum}}\}_{\mathcal{S}\text{-partition } \mathcal{P} \text{ of } X}$ of a measurable $X \xrightarrow{f} \overline{\mathbb{R}}_{\geq 0}$ on the measure space $(X, \mathcal{S}, \mu)$[ix]

2.1 $\int \chi_E\,d\mu = \mu_E$ $\forall$*measure space* $(X, \mathcal{S} \ni E, \mu)$

Proof. $\int \chi_E\,d\mu \geq \mathcal{L}_{\chi_E, \exists\mathcal{S}\text{-partition } \{E, X\setminus E\} \text{ of } X} = \mu_E \geq \mu_{\biguplus_{j=1, A_j\subseteq E}^m A_j} = \sum_{j=1, A_j\subseteq E}^m \mu_{A_j}$

$$= \sum_{j=1}^m \left(\mu_{A_j} \inf_{A_j} \chi_E = \begin{cases} \mu_{A_j} & \text{if } A_j \subseteq E \\ 0 & \text{if } A_j\setminus E \neq \varnothing \end{cases}\right) = \mathcal{L}_{\chi_E, \forall\mathcal{S}\text{-partition } \{A_j\}_{j=1}^m \text{ of } X}$$ □

E.g. $\forall$ Lebesgue's measure $\mathring{\mu}$ on X, $\int \chi_{\mathbb{Q}}\,d\mathring{\mu} = \mathring{\mu}_{\mathbb{Q}} = 0$, $\int \chi_{[0,1]\setminus\mathbb{Q}}\,d\mathring{\mu} = \mathring{\mu}_{[0,1]\setminus\mathbb{Q}} = 1$.

E.g. $\int b\,d\mu = \sum_{k=1}^\infty b_k$, with $\mathbb{Z}_{>0} \xrightarrow{b:k\mapsto b_k} \mathbb{R}_{\geq 0}$, and μ the counting measure on $\mathbb{Z}_{>0}$.[x]

2.2 $\int \left(\sum_{k=1}^n c_k \chi_{E_k}\right) d\mu \overset{\forall\{c_k\in\overline{\mathbb{R}}_{\geq 0}\}_{k=1}^n}{\underset{\forall \textit{disjoint sequence } \{E_k\in\mathcal{S}\}_{k=1}^n}{=\!=\!=}} \sum_{k=1}^n c_k \mu_{E_k}$ $\forall$*measure space* $(X, \mathcal{S}, \mu)$

Proof. Define $f := \sum_{k=0}^n c_k \chi_{E_k}$, with $\{E_0 := X\setminus\biguplus_{k=1}^n E_k\} \cup \{E_k\}_{k=1}^n$ an $\mathcal{S}$-partition of X, and set $c_0 \equiv 0$. Then

$$\sum_{k=1}^n c_k \mu_{E_k} \equiv \sum_{k=0}^n c_k \mu_{E_k} = \mathcal{L}_{f,\{E_k\}_{k=0}^n} \leq \int \left(f \equiv \sum_{k=1}^n c_k \chi_{E_k}\right) d\mu$$
$$= \mathcal{L}_{f, \exists\mathcal{S}\text{-partition } \{A_j = [B_j = \biguplus_{k=1}^n (B_{j,k} = A_j \cap E_k)] \uplus [A_j\setminus B_j]\}_{j=1}^m \text{ of } X}$$
$$= \sum_{j=1}^m \left\{\left[\mu_{A_j} \overset{\mu_\varnothing = 0}{=\!=} \begin{cases} \sum_{k=1, B_{j,k}\neq\varnothing}^n \mu_{B_{j,k}} + \mu_{A_j\setminus B_j} & \text{if } A_j\setminus B_j \neq \varnothing \\ \sum_{k=1, B_{j,k}\neq\varnothing}^n \mu_{B_{j,k}} & \text{if } A_j\setminus B_j = \varnothing \end{cases}\right]\right.$$
$$\left.\times\left[\inf_{A_j} f \overset{(A_j\setminus B_j)\cap(E_{\forall k\in\{1,\dots,n\}} = \biguplus_{j=1}^m B_{j,k}) = \varnothing}{=\!=\!=} \begin{cases} 0 & \text{if } A_j\setminus B_j \neq \varnothing \\ \min_{i\in\{1,\dots,n\}, B_{j,i}\neq\varnothing} c_i & \text{if } A_j\setminus B_j = \varnothing \end{cases}\right]\right\}$$
$$= \sum_{j=1, A_j\setminus B_j=\varnothing}^m \left[\left(\sum_{k=1, B_{j,k}\neq\varnothing}^n \mu_{B_{j,k}}\right) \min_{i\in\{1,\dots,n\}, B_{j,i}\neq\varnothing} c_i \leq \sum_{k=1, B_{j,k}\neq\varnothing}^n \mu_{B_{j,k}} c_k\right]$$
$$\leq \sum_{j=1}^m \sum_{k=1}^n \mu_{B_{j,k}} c_k = \sum_{k=1}^n c_k \left(\sum_{j=1}^m \mu_{B_{j,k}} = \mu_{\biguplus_{j=1}^m B_{j,k} = E_k}\right)$$ □

2.3 $\int f\,d\mu \leq \int g\,d\mu$ $\forall X \xrightarrow{f,\,g} \overline{\mathbb{R}}_{\geq 0}$ *measurable on a measure space* $(X, \mathcal{S}, \mu) : f_{\forall x\in X} \leq g_x$ $(\Rightarrow \inf_{A_{\forall j\in\{1,\dots,m\}}} f \leq \inf_{A_j} g \Rightarrow \mathcal{L}_{f,\mathcal{P}} \leq \mathcal{L}_{g,\mathcal{P}}$, $\forall\mathcal{S}$*-partition* $\mathcal{P} = \{A_j\}_{j=1}^m$ *of* $X)$

2.1.2 Monotone convergence theorem about limits & integrals

2.4 $\int f\,d\mu = \sup_{S=\left\{\sum_{j=1}^m (c_j\in\mathbb{R}_{\geq 0})\mu_{A_j\in\mathcal{S}} \,\middle|\, A_{j=1,\dots,m} \textit{ are disjoint} \wedge f_{\forall x\in X} \geq \sum_{j=1}^m c_j \chi_{A_j;x}\right\}}$

[ix] $\infty \cdot 0 := 0 =: 0 \cdot \infty$

[x] The ***counting measure*** μ on a measurable space $(X, \mathcal{S})$ counts the number of elements in $E \in \mathcal{S}$; *i.e.* $\mu_E := |E|$

$\forall$ *measurable* $X \xrightarrow{f} \overline{\mathbb{R}}_{\geq 0}$ *on a measure space* $(X, \mathcal{S}, \mu)$ ■

Proof. 1. $\underbrace{\int f \, d\mu \geq \overbrace{\int \left(\sum_{j=1}^m c_j \chi_{A_j}\right) d\mu}}_{\text{theorem 2.3}}^{\text{theorem 2.2}} = \sum_{j=1}^m c_j \mu_{A_j}$.

2. *C.f.* definition 2.1. (a) $\inf_{\forall A \in \mathcal{S}: \mu_A > 0} f < \infty \Rightarrow$ '$\forall \mathcal{S}$-partition $\mathcal{P} = \{A_j \in \mathcal{S} \backslash \{\varnothing\}\}_{j=1}^m$ of X, taking $c_j = \inf_{A_j} f$ shows that $\mathcal{L}_{f,\mathcal{P}} \in S \xRightarrow{\text{definition of } \int f d\mu} \sup_S \geq \int f \, d\mu$'.
(b) $\inf_{\exists A \in \mathcal{S}: \mu_A > 0} f = \infty \Rightarrow$ '$\forall t \in \mathbb{R}_{>0}$, taking $\{A_j\}_{j=1}^{m=1} = \{A\}$ and $c_1 = t$ shows that $\sup_S \geq t\mu_A = \infty \geq \int f \, d\mu$' □

Theorem (monotone convergence) $\forall \left\{X \xrightarrow{f_k} \overline{\mathbb{R}}_{\geq 0} \;\middle|\; f_k \leq f_{k+1} \wedge f_{k;\forall x \in X} \xrightarrow{k \to \infty} f_x\right\}_{k=1}^{\infty}$ *of measurable maps on a measure space* $(X, \mathcal{S}, \mu)$, $\int f_k \, d\mu \xrightarrow{k \to \infty} \int f \, d\mu$ ■

Proof. 1. Theorem 1.13 $\Rightarrow X \xrightarrow{f} \overline{\mathbb{R}}_{\geq 0}$ is measurable $\xRightarrow[\text{theorem 2.3}]{f_{\forall k \in \mathbb{Z}_{>0}; \forall x \in X} \leq f_x} \int f_{\forall k \in \mathbb{Z}_{>0}} \, d\mu \leq \int f \, d\mu \Rightarrow \lim_{k \to \infty} \int f_k \, d\mu \leq \int f \, d\mu$

2. $\forall \{c_j \in \mathbb{R}_{\geq 0}\}_{j=1}^m \forall \left\{A_j \in \mathcal{S} \;\middle|\; A_{j=1,\ldots,m} \text{ are disjoint} \wedge f_{\forall x \in X} \geq \sum_{j=1}^m c_j \chi_{A_j;x}\right\}_{j=1}^m$ $\forall t \in (0,1)$, $E_{k \in \mathbb{Z}_{>0}} := \left\{x \in X \;\middle|\; f_{k;x} > t \sum_{j=1}^m c_j \chi_{A_j;x} \wedge \bigcup_{j \in \mathbb{Z}_{>0}} E_j = X\right\} \subseteq E_{k+1} \in \mathcal{S}$ $\xRightarrow{\text{theorem 1.14}} \mu_{A_j \cap E_k} \xrightarrow{k \to \infty} \mu_{A_j}$. Then $f_{\forall k \in \mathbb{Z}_{>0}; \forall x \in X} \geq t \sum_{j=1}^m c_j \chi_{A_j \cap E_k; x}$ $\xRightarrow{\text{theorem 2.4}} \int f_{\forall k \in \mathbb{Z}_{>0}} \, d\mu \geq t \sum_{j=1}^m c_j \mu_{A_j \cap E_k} \xRightarrow{k \to \infty} \lim_{k \to \infty} \int f_k \, d\mu \geq t \sum_{j=1}^m c_j \mu_{A_j}$ $\xrightarrow{t \to 1} \sum_{j=1}^m c_j \mu_{A_j} \xrightarrow{\text{taking supremum over } S \text{ in theorem 2.4}} \int f \, d\mu$ □

2.5 $\forall$ *measure space* $(X, \mathcal{S}, \mu)$, $f = \sum_{j=1}^m \left(a_j \in \overline{\mathbb{R}}_{\geq 0}\right) \chi_{A_j \in \mathcal{S}} = \sum_{k=1}^n \left(b_k \in \overline{\mathbb{R}}_{\geq 0}\right) \chi_{B_k \in \mathcal{S}} = g$ $\Rightarrow \sum_{j=1}^m a_j \mu_{A_j} = \sum_{k=1}^n b_k \mu_{B_k}$ ■

Proof. 1. Say $\bigcup_{j=1}^m A_j = X$.[xi] $\forall$nondisjoint pairs $A'_{k=1,2} \in \{A_j\}_{j=1}^m$, repeat the decomposition

$$\begin{cases} \bigcup_{j=1}^2 A'_j = \underbrace{(A'_1 \backslash A'_2) \uplus \overbrace{(A'_1 \cap A'_2)}}_{A'_1}^{A'_2} \uplus (A'_2 \backslash A'_1) \\ \sum_{j=1}^2 a_j \chi_{A'_j} = a_1 \chi_{A'_1 \backslash A'_2} + (a_1 + a_2) \chi_{A'_1 \cap A'_2} + a_2 \chi_{A'_2 \backslash A'_1} \\ \sum_{j=1}^2 a_j \mu_{A'_j} = a_1 \mu_{A'_1 \backslash A'_2} + (a_1 + a_2) \mu_{A'_1 \cap A'_2} + a_2 \mu_{A'_2 \backslash A'_1} \end{cases}$$

for finite steps, one can convert the initial sets A into disjoint ones with modified coefficients a but unchanged value of '$\sum a\mu_A$'.

2. Replace the sets A corresponding to each modified a from step 1 by $\bigcup A$, μ's finite additivity $\Rightarrow$ '$\sum a\mu_A$'s value remains unchanged when making the coefficients a distinct.

3. Drop any terms for which $A = \varnothing$, getting f's standard[xii] representation with '$\sum a\mu_A$'s value unchanged. Finally, applying the same procedure to g shows that $f = g$ iff $\sum a\mu_A = \sum b\mu_B$. □

[xi] Otherwise add the term $0 \cdot \chi_{X \backslash \bigcup_{j=1}^m A_j}$ to the simple map $X \xrightarrow{f} \overline{\mathbb{R}}_{\geq 0}$

[xii] The representation $\sum_{k=1}^n c_k \chi_{E_k}$ $\forall$simple map $X \xrightarrow{h} \overline{\mathbb{R}}_{\geq 0}$ on a measurable space $(X, \mathcal{S})$ is ***standard*** if $c_{k=1,\ldots,n} \in \overline{\mathbb{R}}_{\geq 0}$ are disjoint $\wedge \left\{E_k = h^{-1}_{\{c_k\}} \neq \varnothing\right\}_{k=1}^n$ is an $\mathcal{S}$-partition of X

2.6 $\int\left(\sum_{k=1}^{n} c_k \chi_{E_k}\right) d\mu \overset{\forall\{c_k \in \overline{\mathbb{R}}_{\geq 0}\}_{k=1}^{n}}{\underset{\forall\{E_k \in \mathcal{S}\}_{k=1}^{n}}{=\!=\!=}} \sum_{k=1}^{n} c_k \mu_{E_k}$ *∀measure space* $(X, \mathcal{S}, \mu)$ ■

Proof. Apply theorems 2.2 & 2.5 on the standard representation of $\sum_{k=1}^{n} c_k \chi_{E_k}$ □

2.7 $\int (f+g)\, d\mu = \int f\, d\mu + \int g\, d\mu$ *∀measurable* $X \xrightarrow{f,g} \overline{\mathbb{R}}_{\geq 0}$ *on a measure space* $(X, \mathcal{S}, \mu)$ ■

Proof. Theorem 1.20 ⇒ ∃increasing sequences $\left\{X \xrightarrow{f_k} \overline{\mathbb{R}}_{\geq 0}\right\}_{k=1}^{\infty}$ & $\left\{X \xrightarrow{g_k} \overline{\mathbb{R}}_{\geq 0}\right\}_{k=1}^{\infty}$ of simple maps measurable on $(X, \mathcal{S}, \mu)$: $f_{\forall x \in X} = f_{k \to \infty; x}$ & $g_{\forall x \in X} = g_{k \to \infty; x}$. Then $\int (f+g)\, d\mu \xleftarrow[\text{monotone convergence theorem}]{\infty \leftarrow k} \int (f_k + g_k)\, d\mu \overset{\text{theorem 2.6}}{=\!=\!=} \int f_k\, d\mu + \int g_k\, d\mu$ $\xrightarrow[\text{monotone convergence theorem}]{k \to \infty} \int f\, d\mu + \int g\, d\mu$ □

2.1.3 Integration of real-valued maps

2.3 Define $X \xrightarrow{f^{\pm}} \overline{\mathbb{R}}_{\geq 0}$ by $f^{\pm}_{\forall x \in X} := \max\{f_x, 0\}$ $\forall X \xrightarrow{f} \overline{\mathbb{R}}$. f is measurable on a measure space $(X, \mathcal{S}, \mu)$ with at least one of $\int f^{\pm}\, d\mu < \infty \Rightarrow \int f\, d\mu := \int f^{+}\, d\mu - \int f^{-}\, d\mu$ ●

Remark. • $\int \left(\left|f = f^{+} - f^{-}\right| = f^{+} + f^{-}\right) d\mu < \infty$ iff $\int \left|f^{\pm}\right| d\mu < \infty$.

• $\int f\, d\mu$ is defined ⇒ measurable f with at least one of $\int f^{\pm}\, d\mu < \infty$.

E.g. $\int \operatorname{sgn} d\mu$ is not defined ∀Lebesgue's measure μ on $\mathbb{R}$ because $\int \operatorname{sgn}^{\pm} d\mu = \infty$.

2.8 *∀measurable* $X \xrightarrow{f} \overline{\mathbb{R}}$ *on a measure space* $(X, \mathcal{S}, \mu)$, $\int f\, d\mu$ *is defined*

$\Rightarrow \int cf\, d\mu \overset{\forall c \in \mathbb{R}}{=\!=\!=} c \int f\, d\mu \wedge \left|\int f\, d\mu\right| \leq \int |f|\, d\mu$ ■

Proof. 1. Without loss of generality, say $c \geq 0$. Then $\int cf\, d\mu = \sum_{s=\pm} s \int (cf)^s\, d\mu$ $\overset{\mathcal{L}_{cg,\mathcal{P}} = c\mathcal{L}_{g,\mathcal{P}} \Rightarrow \int cg d\mu = c \int g d\mu}{\underset{\forall X \xrightarrow{g} \overline{\mathbb{R}}_{\geq 0}\ \forall \text{partition } \mathcal{P} \text{ of } X}{=\!=\!=\!=\!=}} \sum_{s=\pm} s\left(\int cf^s\, d\mu = c \int f^s\, d\mu\right) = c \int f\, d\mu.$

2. $\left|\int f\, d\mu\right| = \left|\sum_{s=\pm} s \underbrace{\int f^s\, d\mu}_{<\infty \text{ for at least one of } s}\right| \leq \sum_{s=\pm} s \int f^s\, d\mu = \int |f|\, d\mu$ □

2.9 *∀measurable* $X \xrightarrow{f_{k=1,2}} \overline{\mathbb{R}}$ *on a measure space* $(X, \mathcal{S}, \mu)$

- $\int \left|f_{k=1,2}\right| d\mu < \infty \Rightarrow \int \left(\sum_{k=1}^{2} f_k\right) d\mu = \sum_{k=1}^{2} \int f_k\, d\mu$ *(c.f. theorem 2.7)*
- $f_{1;\forall x \in X} \leq f_{2;x} \Rightarrow \int f_1\, d\mu \leq \int f_2\, d\mu$ *(c.f. theorem 2.3)* ■

2.2 Limits of integrals & integrals of limits

2.2.1 Bounded convergence theorem

2.10 $\left|\int_E f\, d\mu := \int \chi_E f\, d\mu\right| \leq \int \chi_E \left(|f| \leq \sup_E |f|\right) d\mu = \mu_E \sup_E |f|$ *∀measurable* $X \xrightarrow{f} \overline{\mathbb{R}}$ *on a measure space* $(X, \mathcal{S} \ni E, \mu)$ ■

Theorem (bounded convergence) $\forall \left\{X \xrightarrow{f_k} \mathbb{R} \;\middle|\; f_{k;\forall x\in X} \xrightarrow{k\to\infty} f_x\right\}_{k=1}^{\infty}$ *of measurable maps on a measure space* $(X, \mathcal{S}, \mu)$ *with* $\mu_X < \infty$, $\int f_k \, d\mu \xrightarrow{k\to\infty} \int f \, d\mu$ *if* $\exists c \in \mathbb{R}_{>|f_{\forall k\in\mathbb{Z}_{>0};\forall x\in X}|}$ ■

Proof. Theorem 1.12 $\Rightarrow$ measurable $X \xrightarrow{f} \mathbb{R} \xRightarrow{\text{Egorov's theorem}} \forall \epsilon > 0 \; \exists E \in \mathcal{S}:$ $\mu_{X\setminus E} < {}^{\epsilon}/_{4c} \wedge \{f_k\}_{k=1}^{\infty}$ converges to f uniformly[xiii] on E

$$\Rightarrow \lim_{k\to\infty} \left| \int f_k \, d\mu - \int f \, d\mu = \int_{X\setminus E} f_k \, d\mu - \int_{X\setminus E} f \, d\mu + \int_E (f_k - f) \, d\mu \right|$$

$$\leq \lim_{k\to\infty} \overbrace{\int_{X\setminus E} |f_k| \, d\mu}^{\leq \mu_{X\setminus E}c < {}^{\epsilon}/_4} + \overbrace{\int_{X\setminus E} |f| \, d\mu}^{\leq \mu_{X\setminus E}c < {}^{\epsilon}/_4} + \underbrace{\lim_{k\to\infty} \int_E |f_k - f| \, d\mu}_{\leq (\mu_E < \infty)(\sup_E |f_{k\to\infty} - f| < {}^{\epsilon}/_{2\mu_E})} < \epsilon \xrightarrow{\text{arbitrariness of } \epsilon} 0$$

Remark. Egorov's theorem is crucial for interchanging limits and integrals in proofs.

2.2.2 0-measure sets in integration theorems

2.4 $\forall$measure space $(X, \mathcal{S}, \mu)$, $E \in \mathcal{S}$ contains ***almost every*** $x \in X$ (denote $\underset{\sim}{\forall} x \in X$) if $\mu_{X\setminus E} = 0$ ●

Remark 1. Integration theorems can almost always be relaxed to hold for almost everywhere instead of everywhere. *E.g.* relax in the bounded convergence theorem '$f_{k;\forall x\in X} \xrightarrow{k\to\infty} f_x$' to '$f_{k;\underset{\sim}{\forall} x\in X} \xrightarrow{k\to\infty} f_x$'; *i.e.* $\exists E \in \mathcal{S} : \mu_{X\setminus E} = 0 \wedge f_{k;\forall x\in E} \xrightarrow{k\to\infty} f_x$, then $\int f_k \, d\mu = \int_E f_k \, d\mu \equiv \int \chi_E \left(f_k \xrightarrow{k\to\infty} f\right) d\mu \equiv \int_E f \, d\mu = \int f \, d\mu$.

2.2.3 Dominated convergence theorem

2.11 $\forall$*measurable* $X \xrightarrow{g} \overline{\mathbb{R}}_{\geq 0}$ *on a measure space* $(X, \mathcal{S}, \mu)$ *with* $\int g \, d\mu < \infty \; \forall \epsilon > 0$

1. $\exists \delta > 0 : \int_{\forall B\in\mathcal{S}:\mu_B<\delta} g \, d\mu < \epsilon$

2. $\exists E \in \mathcal{S} : \int_{X\setminus E:\mu_E<\infty} g \, d\mu < \epsilon$ ■

Proof. 1. Theorem 2.4 $\Rightarrow$ '$\exists$simple $\mathcal{S}$-measurable $X \xrightarrow{h\in[0,g]} \mathbb{R}_{\geq 0} : \underbrace{\int g \, d\mu}_{<\infty} - \int h \, d\mu \in [0, {}^{\epsilon}/_2)$' $\Rightarrow \exists \delta > 0 : \delta \underbrace{\max_{\{h_x \mid x\in X\}}}_{H} < {}^{\epsilon}/_2 \wedge \int_{B:\mu_B<\delta} g \, d\mu = \underbrace{\int_B (g-h) \, d\mu}_{\leq \int (g-h)d\mu < {}^{\epsilon}/_2} + \underbrace{\int_B h \, d\mu}_{\leq H\mu_B < H\delta < {}^{\epsilon}/_2} < \epsilon$.

2. '$\exists\mathcal{S}$-measurable partition $\mathcal{P} = \{A_j\}_{j=1}^{m}$ of $X : \underbrace{\int g \, d\mu}_{<\infty} - \mathcal{L}_{g,\mathcal{P}} \in [0, \epsilon) \wedge \mu_{E=\bigcup_{j=1,\ldots,m \;\inf_{A_j} g>0} A_j}$ $< \infty \; (\Leftarrow \mathcal{L}_{g,\mathcal{P}} < \infty) \wedge \inf_{\forall A\in\mathcal{P}:A\not\subseteq E} g = 0 \; (\Rightarrow \mathcal{L}_{g,\mathcal{P}} = \mathcal{L}_{\chi_E g,\mathcal{P}})$'
$\Rightarrow \int_{X\setminus E} g \, d\mu = \int g \, d\mu - \int \chi_E g \, d\mu - \int \chi_E g \, d\mu < \epsilon + \mathcal{L}_{g,\mathcal{P}} - \mathcal{L}_{\chi_E g,\mathcal{P}} = \epsilon$ □

Theorem (dominated convergence) $\forall$*measurable* $X \xrightarrow{f} \overline{\mathbb{R}}$ *on a measure space* $(X, \mathcal{S}, \mu)$ $\forall \left\{\text{measurable } X \xrightarrow{f_k} \overline{\mathbb{R}} \;\middle|\; f_{k;\underset{\sim}{\forall} x\in X} \xrightarrow{k\to\infty} f_x\right\}_{k=1}^{\infty}$, $\int f_k \, d\mu \xrightarrow{k\to\infty} \int f \, d\mu$ *if* $\exists$*measurable* $X \xrightarrow{g} \overline{\mathbb{R}}_{\geq 0} : \int g \, d\mu < \infty \wedge \left|f_{\forall k\in\mathbb{Z}_{>0};\underset{\sim}{\forall} x\in X}\right| \leq g_x$ ■

[xiii]*I.e.* $|f_k - f|$ arbitrarily small for large enough k

Proof. $\left|\int f_k\,d\mu - \int f\,d\mu\right| \overset{\forall E\in\mathcal{S}}{=\!=} \left|\int_{X\setminus E} f_k\,d\mu - \int_{X\setminus E} f\,d\mu + \int_E f_k\,d\mu - \int_E f\,d\mu\right|$
$\leq \left(\left|\int_{X\setminus E} f_k\,d\mu\right| + \left|\int_{X\setminus E} f\,d\mu\right| \leq \left|2\int_{X\setminus E} g\,d\mu\right|\right) + \left|\int_E (f_k - f)\,d\mu\right|$

1. $\mu_X < \infty \xRightarrow{\text{Egorov's theorem}} \exists E \in \mathcal{S} : \mu_{X\setminus E\in\mathcal{S}} < \infty \, (\xRightarrow{\text{theorem 2.11.1}} \int_{X\setminus E} g\,d\mu < {}^{\epsilon}\!/_4) \wedge \{f_k\}_{k=1}^{\infty}$ converges uniformly on E to f ($\Rightarrow \left|\int_E (f_k - f)\,d\mu\right| < {}^{\epsilon}\!/_2$ for large enough k). Thus $\left|\int f_k\,d\mu - \int f\,d\mu\right| \xrightarrow{k\to\infty} 0$

2. For $\mu_X = \infty$, theorem 2.11.2 $\Rightarrow \exists E \in \mathcal{S} : \mu_E < \infty \wedge \int_{X\setminus E} g\,d\mu < {}^{\epsilon}\!/_4$. Besides, $\left|\int_E f_k\,d\mu - \int_E f\,d\mu\right| < {}^{\epsilon}\!/_2$ for large enough k by case 1 as applied to $\left\{f_k|_E\right\}_{k=1}^{\infty}$. Thus $\left|\int f_k\,d\mu - \int f\,d\mu\right| \xrightarrow{k\to\infty} 0$ □

2.2.4 Riemann's & Lebesgue's integrals

2.12 *A bounded* $[a,b] \xrightarrow{f} \mathbb{R}$ *is* Riemann-*integrable iff* $\mathring{\mu}_{\{x\in[a,b]\,|\,f\text{ is discontinuous at }x\}} = 0$ *(say* $-\infty < a < b < \infty$*); besides,* f *is measurable on the measure space* $(\mathbb{R}, \mathscr{L}, \mathring{\mu})$*, with* Riemann*'s integral* $\int_a^b f = \int_{[a,b]} f\,d\mathring{\mu}$[xiv] ■

Proof. $\forall$partition $\mathcal{P}_{\forall n\in\mathbb{Z}_{>0}}$ dividing $[a,b]$ into 2^n subintervals $I_{j=1,\ldots,2^n}$ of equal size ${}^{(b-a)}\!/_{2^n}$, Riemann's lower sum $L_{f,\mathcal{P}_n,[a,b]} = \int_{[a,b]}\left(g_n = \sum_{j=1}^{2^n} \chi_{I_j} \inf_{I_j} f\right) d\mathring{\mu}$ & upper sum $U_{f,\mathcal{P}_n,[a,b]} = \int_{[a,b]}\left(h_n = \sum_{j=1}^{2^n} \chi_{I_j} \sup_{I_j} f\right) d\mathring{\mu}$.[xv] Then $g_1 \leq \cdots \leq g_{n\to\infty} \leq f \leq h_{n\to\infty} \leq \cdots \leq h_1$ $\xRightarrow[\text{(if applicable; c.f. remark 1)}]{\text{bounded convergence theorem}}$ Riemann's lower & upper integrals $L_{f,[a,b]}$ $= \lim_{n\to\infty} L_{f,\mathcal{P}_n,[a,b]} = \int_{[a,b]} g_{n\to\infty}\,d\mathring{\mu}$ & $U_{f,[a,b]} = \lim_{n\to\infty} U_{f,\mathcal{P}_n,[a,b]} = \int_{[a,b]} h_{n\to\infty}\,d\mathring{\mu}$.
Thus Riemann-integrable f $\xLeftrightarrow{\text{by definition}} L_{f,[a,b]} \equiv U_{f,[a,b]}$
$\Longleftrightarrow \int_{[a,b]} \left(h_{n\to\infty} - g_{n\to\infty} \geq 0\right) d\mathring{\mu} = 0$
$\Longleftrightarrow 0 = \left\{x \in [a,b] \,\middle|\, g_{n\to\infty;x} \neq h_{n\to\infty;x}\right\} = \{x \in [a,b] \mid f \text{ is discontinuous at } x\}$ □

2.2.5 Appoximation by nice maps

2.5 $\forall$measurable $X \xrightarrow{f} \overline{\mathbb{R}}$ on a measure space $(X, \mathcal{S}, \mu)$, f's $\mathcal{L}^1$***-norm*** $\|f\|_1 := \int |f|\,d\mu$;
Lebesgue*'s space* $\mathcal{L}^1_\mu := \left\{\mathcal{S}\text{-measurable } X \xrightarrow{f} \mathbb{R} \,\middle|\, \|f\|_1 < \infty\right\}$ ●

E.g. $\forall$measure space $(X, \mathcal{S}, \mu)$, $f \xlongequal[E_{k=1,\ldots,n}\in X \text{ disjoint}]{a_{k=1,\ldots,n}\in\mathbb{R}_{\neq 0} \text{ distinct}} \sum_{k=1}^n a_k \chi_{E_k} \in \mathcal{L}^1_\mu$ iff $\mu_{E_{\forall k\in\{1,\ldots,n\}\in\mathcal{S}}} < \infty$, with $\|f\|_1 = \sum_{k=1}^n |a_k| \mu_{E_k}$.
E.g. $\mathcal{L}^1_\mu$ is ℓ^1 if μ is the counting measure on the measurable space $(\mathbb{Z}_{>0}, 2^{\mathbb{Z}_{>0}})$.
Say $\mathbb{Z}_{>0} \xrightarrow{a:k\mapsto a_k} \mathbb{R}$, then $\|a \in \ell^1\|_1 = \sum_{k=1}^{\infty} |a_k| < \infty$.
Properties ($\mathcal{L}^1$-norm's) $\forall$measure space $(X, \mathcal{S}, \mu)$ $\forall$f & $g \in \mathcal{L}^1_\mu$

[xiv] Say $-\infty \leq a < b < c \leq \infty$, $(a,b) \xrightarrow{f} \mathbb{R}$ is measurable on $(\mathbb{R}, \mathscr{L}, \mathring{\mu})$, then $-\int_b^a f = \int_a^b f \equiv \int_a^b f_x\,dx \equiv \int_{(a,b)} f\,d\mathring{\mu} = \int_a^c f + \int_c^b f$

[xv] For aesthetically pleasing form of mathematics, at each of the endpoints (other than a & b) that is in two of the subintervals, change g_n's value to be f's infimum over the two subintervals, and h_n's value to be f's supremum over the two subintervals.

- $\|f\|_1 \geq 0$
- $\|f\|_1 = 0$ iff $f_{\underset{\sim}{\forall} x \in X} = 0$
- $\|cf\|_1 \overset{\forall c \in \mathbb{R}}{=\!=} |c| \cdot \|f\|_1$
- $\left\|f + g\right\|_1 \leq \|f\|_1 + \left\|g\right\|_1$
- $\forall \epsilon > 0\, \exists$simple $h \in \mathcal{L}^1_\mu : \|f - h\|_1 < \epsilon$ ■

2.6 Denotes $\mathcal{L}^1_{\mathring{\mu}}$ by $\mathcal{L}^1_{\mathbb{R}}$ for the measure space $\left(\mathbb{R}, \mathscr{F} \in \{\mathscr{B}, \mathscr{L}\}, \mathring{\mu}\right)$, with $\|f\|_1 = \int_{\mathbb{R}} |f|\, d\mathring{\mu}$ ●

2.7 $\mathbb{R} \xrightarrow{\vartheta = \sum_{k=1}^n a_k \chi_{I_k}} \mathbb{R}$ with intervals $I_{k=1,\ldots,n} \subseteq \mathbb{R}$ and $a_{k=1,\ldots,n} \in \mathbb{R}_{\neq 0}$ is a ***step map*** ●

Remark. • $\|\vartheta\|_1 = \sum_{k=1}^n \left|a_k\right| \mathring{\mu}_{I_k}$ if $I_{k=1,\ldots,n}$ are disjoint.

- $\vartheta \in \mathcal{L}^1_{\mathbb{R}}$ iff $\mathring{\mu}_{I_{\forall k \in \{1,\ldots,n\}}} < \infty$.
- The intervals in ϑ's definition can be open or closed, or half-open; including/excluding interval endpoints does not matter when using ϑ in integrals.

2.13 $\forall f \in \mathcal{L}^1_{\mathbb{R}}\, \forall \epsilon > 0$

- $\exists$*step* $\vartheta \in \mathcal{L}^1_{\mathbb{R}} : \|f - \vartheta\|_1 < \epsilon$
- $\exists$*continuous* $\mathbb{R} \xrightarrow{g} \mathbb{R} : \left\|f - g\right\|_1 < \epsilon \wedge \mathring{\mu}_{\{x \in \mathbb{R} \mid g_x \neq 0\}} < \infty$ ■

3 Differentiation

3.1 Hardy-Littlewood's maximal map

Inequality (Markov's) $\mu_{\{x\in X \mid |h(x)|>c\}} \leq \|h\in\mathcal{L}^1_\mu\|_1 / (c>0)$ $\forall$ *measure spaces* $(X, \mathcal{S}, \mu)$ ■

Lemma (Vitali's covering) *Every sequence* $\{I_k \subseteq \mathbb{R}\}_{k=1}^n$ *of bounded nonempty open intervals has a disjoint subsequence* $\{I_{k_j}\}_{j=1}^m : \bigcup_{k=1}^n I_k \subseteq \bigcup_{j=1}^m 3I_{k_j}$, *with* $3I$ *the open interval with the same centre as* I *and* $\mathring{\mu}_{3I} = 3\mathring{\mu}_I$ ■

Inequality (Hardy-Littlewood's maximal) $\mathring{\mu}_{\{b\in\mathbb{R} \mid h^*_b>c\}} \leq 3\|h\in\mathscr{L}^1_\mathbb{R}\|_1 / (c>0)$, *with* $\mathbb{R} \xrightarrow{h^* : b \mapsto \sup_{t>0} \left(\int_{b-t}^{b+t} |h|\right) / 2t} \overline{\mathbb{R}}_{\geq 0}$ ***Hardy-Littlewood's maximal map***[xvi] $\forall \mathscr{L}$-*measurable* $\mathbb{R} \xrightarrow{h} \mathbb{R}$ ■

3.2 Derivatives of integrals

3.1 ($I \xrightarrow{g} \mathbb{R}$)'s ***derivative*** $g'_b := \lim_{t\to 0} (g_{b+t}-g_b)/t$ (if the limit exists; g is then dubbed ***differentiable***) at $b \in I$ $\forall$ open interval $I \subseteq \mathbb{R}$ ●

Fundamental theorem of calculus $f \in \mathscr{L}^1_\mathbb{R}$ *is continuous at* $b \in \mathbb{R} \Rightarrow g'_b = f_b$ *with* $\mathbb{R} \xrightarrow{g:x\mapsto\int_{-\infty}^x f} \mathbb{R}$

Theorem (Lebesgue's differentiation) $f \in \mathscr{L}^1_\mathbb{R} \Rightarrow \underset{\sim}{\forall} b \in \mathbb{R}$

- $\lim_{t\downarrow 0} \left(\int_{b-t}^{b+t} |f-f_b|\right) / 2t = 0$
- $g'_b = f_b$ *with* $\mathbb{R} \xrightarrow{g:x\mapsto\int_{-\infty}^x f} \mathbb{R}$ ■

3.1 $\nexists \mathscr{L}$-*measurable* $E \subseteq [0,1] : \mathring{\mu}_{E\cap[0,b]} = b/2\ \forall b \in [0,1]$ ■

Proof. $\exists$such $E \Rightarrow g_{b\in\mathbb{R}} = \int_{-\infty}^b \chi_E \overset{\forall b\in[0,1]}{=\!=} b/2$

$\Rightarrow 1/2 \overset{\forall b\in(0,1)}{=\!=} g'_b \underset{\text{Lebesgue's differentiation theorem}}{\overset{\underset{\sim}{\forall} b\in\mathbb{R}}{=\!=\!=}} \chi_{E;b} \in \{0,1\}$ □

3.2 $f_{\underset{\sim}{\forall} b\in\mathbb{R}} = \lim_{t\downarrow 0} \left(\int_{b-t}^{b+t} f\right) / 2t\ \forall f \in \mathcal{L}^1_\mathbb{R}$ ■

3.2 $\rho_{E\subseteq\mathbb{R};b\in\mathbb{R}} := \lim_{t\downarrow 0} (\mathring{\mu}_{E\cap(b-t,b+t)}) / 2t$ is E's ***density*** at b ●

E.g. $\rho_{[0,1];b} = \begin{cases} 1 & \text{if } b \in (0,1) \\ 1/2 & \text{if } b \in \{0,1\}. \\ 0 & \text{otherwise} \end{cases}$

Theorem (Lebesgue's density) $\rho_{\forall E\in\mathscr{L};b} = \begin{cases} 1 & \underset{\sim}{\forall} b \in E \\ 0 & \underset{\sim}{\forall} b \in \mathbb{R}\setminus E \end{cases}$ ■

3.3 $\exists E \in \mathscr{B} : 0 < \mathring{\mu}_{E\cap I} < \mathring{\mu}_I\ \forall$ *nonempty bounded open interval* I ■

[xvi] *E.g.* $(\chi_{[-1,1]/2})^*_b = \begin{cases} 1/(1+2|b|) & \text{if } 2|b| \geq 1 \\ 1 & \text{if } 2|b| < 1 \end{cases}$

4 Product Measures

4.1 Product of measure spaces

4.1.1 Product σ-algebras

4.1 $A \times B$ is a ***rectangle*** in $X \times Y$ $\forall (A, B) \in 2^{X \times Y}$ ●

4.2 The ***product*** $\mathcal{S} \otimes \mathcal{T}$ is the smallest σ-algebra on $X \times Y$ containing all rectangles $A \times B$ (dubbed ***measurable***) with $(A, B) \in \mathcal{S} \times \mathcal{T}$ $\forall$measurable spaces $(X, \mathcal{S})$ & $(Y, \mathcal{T})$ ●

4.3 $[E]_{a \in X} := \{y \in Y \mid (a, y) \in E\}$ and $[E]^{b \in Y} := \{x \in X \mid (x, b) \in E\}$ are the ***cross sections*** of $E \subseteq X \times Y$ ●

Example 4.1 $[A \times B]_{a \in X} = \begin{cases} B & \text{if } a \in A \\ \varnothing & \text{if } a \notin A \end{cases}$ & $[A \times B]^{b \in Y} = \begin{cases} A & \text{if } b \in B \\ \varnothing & \text{if } b \notin B \end{cases}$ $\forall (A, B) \in 2^{X \times Y}$.

4.1 $([E]^{b \in Y}, [E]_{a \in X}) \in \mathcal{S} \times \mathcal{T}$ $\forall E \in \mathcal{S} \otimes \mathcal{T}$ *$\forall$measurable spaces* $(X, \mathcal{S})$ *&* $(Y, \mathcal{T})$ ■

Proof. $A \times B \in \mathcal{E} = \{E \subseteq X \times Y \mid ([E]^{b \in Y}, [E]_{a \in X}) \in \mathcal{S} \times \mathcal{T}\}$ $\forall (A, B) \in \mathcal{S} \times \mathcal{T}$ by example 4.1, with $\mathcal{E}$ closed under complementation and countable unions as $[(X \times Y) \setminus E]_a = Y \setminus [E]_a$, $[\bigcup_{k \in \mathbb{Z}_{>0}} (E_k \subseteq X \times Y)]_a = \bigcup_{k \in \mathbb{Z}_{>0}} [E_k]_a$ $\forall a \in X$ *etc.* Hence $\mathcal{E}$ is a σ-algebra on $X \times Y$ containing all $A \times B \in \mathcal{S} \otimes \mathcal{T}$; *i.e.* $\mathcal{S} \otimes \mathcal{T} \subseteq \mathcal{E}$ □

4.4 $Y \xrightarrow{[f]_{\forall a \in X} : y \mapsto f_{a,y}} \mathbb{R}$ & $X \xrightarrow{[f]^{\forall b \in Y} x \mapsto f_{x,b}} \mathbb{R}$ are the ***cross sections*** of $X \times Y \xrightarrow{f} \mathbb{R}$ ●

4.2 $[f]_{\forall a \in X}$ *is $\mathcal{T}$-measurable on* Y *and* $[f]^{\forall b \in Y}$ *is $\mathcal{S}$-measurable on* X $\forall \mathcal{S} \otimes \mathcal{T}$*-measurable* $X \times Y \xrightarrow{f} \mathbb{R}$ *$\forall$measurable spaces* $(X, \mathcal{S})$ *&* $(Y, \mathcal{T})$ ■

Proof. $\forall B \in \mathcal{B}$, $\mathcal{S} \otimes \mathcal{T}$-measurable $f \Rightarrow f_B^{-1} \in \mathcal{S} \otimes \mathcal{T} \xRightarrow{\text{theorem 4.1}} [f_B^{-1}]_a \in \mathcal{T}$; besides, $y \in ([f]_a)_B^{-1} \iff f_{a,y} = ([f]_a)_y \in B \iff (a, y) \in f_B^{-1} \iff y \in [f_B^{-1}]_a$. Thus $([f]_a)_{\forall B \in \mathcal{B}}^{-1} = [f_B^{-1}]_a \in \mathcal{T}$; *i.e.* $[f]_a$ is $\mathcal{T}$-measurable. Similarly, $[f]^b$ is $\mathcal{S}$-measurable. □

4.1.2 Monotone class theorem

4.5 $\mathcal{A} \subseteq 2^X$ is an ***algebra*** on X if

- $\varnothing \in \mathcal{A}$
- $E \in \mathcal{A} \Rightarrow X \setminus E \in \mathcal{A}$
- $E_{k=1,2} \in \mathcal{A} \Rightarrow \bigcup_{k=1}^{2} E_k \in \mathcal{A}$ ●

4.3 *$\forall$measurable spaces* $(X, \mathcal{S})$ *and* $(Y, \mathcal{T})$*, the set* $\mathcal{A}$ *of finite unions of rectangles in* $\mathcal{S} \otimes \mathcal{T}$ *is an algebra on* $X \times Y$*, each such union equals a finite union of disjoint measurable rectangles in* $\mathcal{S} \otimes \mathcal{T}$ ■

Proof. 1. (a) Obviously $\mathcal{A}$ is closed under finite unions.

(b) $\forall A_{1,\dots,n}$ & $C_{1,\dots,m} \in \mathcal{S}$ $\forall B_{1,\dots,n}$ & $D_{1,\dots,m} \in \mathcal{T}$, $\left(\bigcup_{j=1}^{n} A_j \times B_j\right) \bigcap \left(\bigcup_{k=1}^{m} C_k \times D_k\right) = \bigcup_{j=1}^{n} \bigcup_{k=1}^{m} [(A_j \times B_j) \cap (C_k \times D_k) = (A_j \cap C_k) \times (B_j \cap D_k)]$; intersection of two rectangles is a rectangle, implying that $\mathcal{A}$ is closed under finite intersections.

(c) $(X \times Y) \setminus (A \times B) = [(X \setminus A) \times Y] \cup [X \times ((Y \setminus B))]$ $\forall (A, B) \in \mathcal{S} \times \mathcal{T}$. Hence the complement of each $\mathcal{S} \otimes \mathcal{T}$-measurable rectangle is in $\mathcal{A}$. Thus the complement of a finite union of $\mathcal{S} \otimes \mathcal{T}$-measurable rectangles is in $\mathcal{A}$ (use DE MORGAN's laws and step (b) that $\mathcal{A}$ is closed under finite intersections). *I.e.* $\mathcal{A}$ is closed under complementation.

2. $[A \times B] \cup [C \times D] = [A \times B] \uplus [C \times (D \backslash B)] \uplus [(C \backslash A) \times (B \cap D)]$ $\forall \mathcal{S} \otimes \mathcal{T}$-measurable rectangles $A \times B$ & $C \times D$. Hence $\forall$finite union of $\mathcal{S} \otimes \mathcal{T}$-measurable rectangles, if it is not a disjoint union, choose any nondisjoint pair of measurable rectangles in the union and replace them with the union of three disjoint measurable rectangles as above. Iterate this process until obtaining a disjoint union of measurable rectangles. □

4.6 $\mathcal{M} \subseteq 2^X$ is a ***monotone class*** on X if

- $\left\{E_k \in \mathcal{M} \mid E_{\forall j \in \mathbb{Z}_{>0}} \subseteq E_{j+1}\right\}_{k \in \mathbb{Z}_{>0}} \Rightarrow \bigcup_{k \in \mathbb{Z}_{>0}} E_k \in \mathcal{M}$
- $\left\{E_k \in \mathcal{M} \mid E_{\forall j \in \mathbb{Z}_{>0}} \supseteq E_{j+1}\right\}_{k \in \mathbb{Z}_{>0}} \Rightarrow \bigcap_{k \in \mathbb{Z}_{>0}} E_k \in \mathcal{M}$ ●

Theorem (monotone class) *The smallest σ-algebra $\mathcal{S}$ containing an algebra $\mathcal{A}$ on X is the smallest monotone class $\mathcal{M}$ containing $\mathcal{A}$* ■

Proof. 1. Every σ-algebra is a monotone class $\Rightarrow \mathcal{M} \subseteq \mathcal{S}$.

2. (a) $A \in \mathcal{A} \Rightarrow \mathcal{A} \subseteq$ monotone class $\mathcal{E} = \{E \in \mathcal{M} \mid A \cup E \in \mathcal{M}\}$ (as $\mathcal{A} \subseteq \mathcal{M}$ is closed under finite union) $\Rightarrow A \cup E \in \mathcal{M} \subseteq \mathcal{E}\ \forall E \in \mathcal{M} \Rightarrow$

(b) $\mathcal{A} \subseteq$ monotone class $\mathcal{D} = \{D \in \mathcal{M} \mid D \cup E \in \mathcal{M}\ \forall E \in \mathcal{M}\} \Rightarrow \mathcal{M} \subseteq \mathcal{D}$ is closed under finite union $\Rightarrow$

(c) i. $F_k = \bigcup_{j=1}^{k} (E_j \in \mathcal{M}) \in \mathcal{M} \Rightarrow F_{k \to \infty} = \bigcup_{k=1}^{\infty} F_k \subseteq \mathcal{M}$ (as $\mathcal{M}$ is a monotone class) $\Rightarrow \mathcal{M}$ is closed under countable union.

ii. $\mathcal{A}$ is closed under complementation $\Rightarrow \mathcal{A} \subseteq$ monotone class $\mathcal{M}'$ $= \{E \in \mathcal{M} \mid X \backslash E \in \mathcal{M}\} \Rightarrow \mathcal{M} \subseteq \mathcal{M}'$ is closed under complementation.

Hence $\mathcal{M}$ is an σ-algebra containing $\mathcal{A}$, and thus $\mathcal{M} \supseteq \mathcal{S}$ □

4.1.3 Products of measures

4.7 A measure μ on a measurable space $(X, \mathcal{S})$ is dubbed

Finite if $\mu_X < \infty$.

σ-finite if $X = \bigcup_{k \in \mathbb{Z}_{>0}} (X_k \in \mathcal{S})$ with $\mu_{X_{\forall k \in \mathbb{Z}_{>0}}} < \infty$ ●

E.g. • Lebesgue's measure on $[0, 1]$ is finite.

• Lebesgue's measure on $\mathbb{R}$ is not finite but σ-finite.

• Counting measure on $\mathbb{R}$ is not σ-finite (because the countable union of finite sets is countable).

4.4 *$\forall \sigma$-finite measure spaces $(X, \mathcal{S}, \mu)$ & $(Y, \mathcal{T}, \nu)$*

1. *$x \mapsto \nu_{[E]_{x \in X}}$ is $\mathcal{S}$-measurable on X and $y \mapsto \mu_{[E]^{y \in Y}}$ is $\mathcal{T}$-measurable on $Y\ \forall E \in \mathcal{S} \otimes \mathcal{T}$.*

2. *the **product** $\mathcal{S} \otimes \mathcal{T} \xrightarrow{\mu \times \nu : E \mapsto \int_X \int_Y \chi_{E;x,y} d\nu_y d\mu_x} (\mu \times \nu)_{\mathcal{S} \otimes \mathcal{T}}$ is a measure on $(X \times Y, \mathcal{S} \otimes \mathcal{T})$*

■

Proof. 1. Without lose of generality, one just need to prove that $x \mapsto \nu_{[E]_x}$ (well-defined, as $[E \in \mathcal{S} \otimes \mathcal{T}]_{\forall x \in X} \in \mathcal{T} \Leftarrow$ theorem 4.1) is $\mathcal{S}$-measurable on X.

(a) If ν is finite, one need to prove that

$$\mathcal{S} \otimes \mathcal{T} = \mathcal{M} = \left\{E \in \mathcal{S} \otimes \mathcal{T} : x \mapsto \nu_{[E]_x} \text{ is } \mathcal{S}\text{-measurable on } X\right\}.$$

By example 4.1, $(A,B) \in \mathcal{S}\times\mathcal{T} \Rightarrow \nu_{[A\times B]_x} = \nu_B\chi_{A;x}\ \forall x\in X$; *i.e.* $x \mapsto \nu_{[A\times B]_x}$ equals the $\mathcal{S}$-measurable map $\nu_B\chi_A$ on X. Hence $\mathcal{M}$ contains all measurable rectangles in $\mathcal{S}\otimes\mathcal{T}$.

By theorem 4.3, $E\ \in$ algebra $\mathcal{A}$ of all finite unions of measurable rectangles in $\mathcal{S}\otimes\mathcal{T} \Rightarrow \exists$measurable rectangles $E_{k=1,\dots,n}$: $\nu_{[E=\uplus_{k=1}^{n} E_k]_x=\uplus_{k=1}^{n}[E_k]_x} = \sum_{k=1}^{n}\nu_{[E_k]_x}$. *I.e.* $x\mapsto \nu_{[E]_x}$ is a finite sum of $\mathcal{S}$-measurable maps and is thus $\mathcal{S}$-measurable. Hence $E\in\mathcal{M}$, and $\mathcal{A}\subseteq\mathcal{M}$.

The next is to show that $\mathcal{M}$ is a monotone class on $X\times Y$. $\forall$increasing sequence $\{E_k\in\mathcal{M}\}_{k=1}^{\infty}$, $\nu_{[\bigcup_{k=1}^{\infty}E_k]_x=\bigcup_{k=1}^{\infty}[E_k]_x} \xleftarrow{\infty\leftarrow k} \nu_{[E_k]_x}$. Hence $x\mapsto\nu_{[\bigcup_{k=1}^{\infty}E_k]_x}$ is $\mathcal{S}$-measurable,[xvii] $\bigcup_{k=1}^{\infty}E_k\in\mathcal{M}$, and $\mathcal{M}$ is closed under countable increasing unions. $\forall$decreasing sequence $\{E_k\in\mathcal{M}\}_{k=1}^{\infty}$, $\nu_{[\bigcap_{k=1}^{\infty}E_k]_x=\bigcap_{k=1}^{\infty}[E_k]_x} \xleftarrow{\infty\leftarrow k} \nu_{[E_k]_x}$ for finite ν. Hence $x\mapsto\nu_{[\bigcap_{k=1}^{\infty}E_k]_x}$ is $\mathcal{S}$-measurable, $\bigcap_{k=1}^{\infty}E_k\in\mathcal{M}$, and $\mathcal{M}$ is closed under countable decreasing intersections.

Finally, monotone class theorem $\Rightarrow$ the monotone class $\mathcal{M}$ containing $\mathcal{A}$ contains the smallest σ-algebra containing $\mathcal{A}$; *i.e.* $\mathcal{M}\supseteq\mathcal{S}\otimes\mathcal{T}$.

(b) If ν is a σ-finite, $\exists\{Y_k\in\mathcal{T}\}_{k=1}^{\infty}$: $\bigcup_{k=1}^{\infty}Y_k = Y \wedge \nu_{Y_{\forall k\in\mathbb{Z}_{>0}}} < \infty$. Replacing each Y_k by $Y_1\cup\cdots\cup Y_k$, one can assume that $Y_1 \subseteq Y_2 \subseteq \cdots$. $\forall E\in\mathcal{S}\otimes\mathcal{T}$, $\nu_{[E]_x} \xleftarrow{\infty\leftarrow k} \nu_{[E\cap(X\times Y_k)]_x}$, with $x\mapsto\nu_{[E\cap(X\times Y_k)]_x}$ $\mathcal{S}$-measurable on X (by step (a), with ν considered finite when restricted to the σ-algebra on Y_k consisting of $\mathcal{T}$-measurable sets $E\subseteq Y_k$). Hence $x\mapsto\nu_{[E]_x}$ is $\mathcal{S}$-measurable on X.

2. Clearly $(\mu\times\nu)_\varnothing = 0$, and $\mu\times\nu$ is the countably additive as $(\mu\times\nu)_{\uplus_{k=1}^{\infty}(E_k\in\mathcal{S}\otimes\mathcal{T})}$
$= \int_X\left(\nu_{[\uplus_{k=1}^{\infty}E_k]_x=\uplus_{k=1}^{\infty}[E_k]_x} = \sum_{k=1}^{\infty}\nu_{[E_k]_x}\right)d\mu_x \overset{\text{monotone convergence theorem}}{=\!=\!=} \sum_{k=1}^{\infty}\int_X\nu_{[E_k]_x}\,d\mu_x$
$= \sum_{k=1}^{\infty}(\mu\times\nu)_{E_k}$ □

E.g. $(\mu\times\nu)_{A\times B} = \mu_A\nu_B\ \forall(A,B)\in\mathcal{S}\times\mathcal{T}$

4.2 Iterated integrals

Theorem (TONELLI'S) $\int_{X\times Y} f\,d(\mu\times\nu) = \int_X\underbrace{\int_Y f_{x,y}\,d\nu_y}_{\mathcal{S}\text{-measurable on }X}d\mu_x = \int_Y\overbrace{\int_X f_{x,y}\,d\mu_x}^{\mathcal{T}\text{-measurable on }Y}d\nu_y\ \forall\mathcal{S}\otimes\mathcal{T}$-*measurable* $X\times Y\xrightarrow{f}\overline{\mathbb{R}}_{\geq 0}$ *on σ-finite measure spaces* $(X,\mathcal{S},\mu)$ & $(Y,\mathcal{T},\nu)$ ■

E.g. Consider $\mathbb{Z}_{>0}\times\mathbb{Z}_{>0}\xrightarrow{x:(j,k)\mapsto x_{j,k}}\overline{\mathbb{R}}_{\geq 0}$ and σ-finite counting measure spaces $(\mathbb{Z}_{>0},2^{\mathbb{Z}_{>0}},\mu)$, then $\int_{\mathbb{Z}_{>0}\times\mathbb{Z}_{>0}}x\,d(\mu\times\mu) = \left(\sum_{j\in\mathbb{Z}_{>0}}\sum_{k\in\mathbb{Z}_{>0}} = \sum_{k\in\mathbb{Z}_{>0}}\sum_{j\in\mathbb{Z}_{>0}}\right)x_{j,k}$.

Theorem (FUBINI'S) $\int_{X\times Y} f\,d(\mu\times\nu) = \int_X\underbrace{\int_Y f_{x,y}\,d\nu_y}_{\mathcal{S}\text{-measurable on }X}d\mu_x = \int_Y\overbrace{\int_X f_{x,y}\,d\mu_x}^{\mathcal{T}\text{-measurable on }Y}d\nu_y\ \forall\mathcal{S}\otimes\mathcal{T}$-*measurable* $X\times Y\xrightarrow{f}\overline{\mathbb{R}}$ *on σ-finite measure spaces* $(X,\mathcal{S},\mu)$ & $(Y,\mathcal{T},\nu)$: $\int_{X\times Y}|f|\,d(\mu\times\nu) < \infty$ *(and thus* $\int_Y\left|f_{\underline{\forall} x\in X,y}\right|d\nu_y < \infty > \int_X\left|f_{x,\underline{\forall} y\in Y}\right|d\mu_x$*)* ■

4.5 $U_f := \{(x,t)\in X\times\mathbb{R}_{>0} \mid 0 < t < f_x\}$ *is the* ***region under the graph*** *of* $X\xrightarrow{f}\overline{\mathbb{R}}_{\geq 0}$.

[xvii]Recall that pointwise limit of $\mathcal{S}$-measurable functions is $\mathcal{S}$-measurable

Then measurable f *on* σ*-finite measure space* $(X, \mathcal{S}, \mu) \Rightarrow U_f \in \mathcal{S} \otimes \mathcal{B}$ $\wedge\, (\mu \times \mathring{\mu})_{U_f} = \int_X f\, d\mu = \int_{\mathbb{R}_{>0}} \mu_{\{x \in X \mid t < f_x\}} d\mathring{\mu}_t$ $\forall$ Lebesgue*'s measure space* $(\mathbb{R}_{>0}, \mathcal{B}, \mathring{\mu})$ ■

4.3 Lebesgue's integrals on $\mathbb{R}^n$

4.6 $\bigtimes_{k=1}^{2} G_k \subseteq \mathbb{R}^{\sum_{k=1}^{2} n_k}$ *is open* $\forall$ *open* $G_{k=1,2} \subseteq \mathbb{R}^{n_k}$ ■

4.8 Borel's $B \subseteq \mathbb{R}^n$ is an element of the smallest σ-algebra on $\mathbb{R}^n$ containing all open $G \subseteq \mathbb{R}^n$; denote the σ-algebra of all Borel's $B \subseteq \mathbb{R}^n$ by $\mathcal{B}_n$ ●

4.7 • $G \subseteq \mathbb{R}^n$ *is open* $\iff G = \bigcup_{k \in \mathbb{Z}_{>0}} C_k$ *with* $C_{\forall k \in \mathbb{Z}_{>0}}$ *open cubes* $\subseteq \mathbb{R}^n$.

• $\mathcal{B}_n$ *is the smallest* σ*-algebra on* $\mathbb{R}^n$ *containing all open cubes* $\subseteq \mathbb{R}^n$ ■

4.8 $\mathcal{B}_{\sum_{k=1}^{2} n_k} = \otimes_{k=1}^{2} \mathcal{B}_{n_k}$ ■

4.9 Define inductively Lebesgue's measure $\mathring{\mu}_n = \mathring{\mu}_{n-1} \times \mathring{\mu}_1$ on measurable spaces $(\mathbb{R}^n, \mathcal{B}_n)$ with $\mathring{\mu}_1$ Lebesgue's measure on $(\mathbb{R}, \mathcal{B}_1)$ ●

4.9 $\forall E \in \mathcal{B}_n\ \forall t \in \mathbb{R}_{>0},\ tE \in \mathcal{B}_n \wedge \mathring{\mu}_{n;tE} = t^n \mathring{\mu}_{n;E}$ ■

4.10 $D_1(D_2 f) = D_2(D_1 f)\ \forall G \xrightarrow{f} \mathbb{R} : \exists$ *continuous* $D_1 f$ & $D_2 f$ & $D_1(D_2 f)$ & $D_2(D_1 f)$ *on the open* $G \subseteq \mathbb{R}^2$, *where the* ***partial derivates*** $(D_1 f)_{x,y} := \lim_{t \to 0} {(f_{x+t,y} - f_{x,y})}/{t}$ & $(D_2 f)_{x,y} := \lim_{t \to 0} {(f_{x,y+t} - f_{x,y})}/{t}\ \forall (x, y) \in G$ *etc.* ■

A Riemann's integration

A.1 Riemann integral

A.1 A ***partition*** of $[a,b] \subseteq \mathbb{R}$ is a finite list $\{x_i\}_{i=0}^n$ with $a = x_0 < x_1 < \cdots < x_n = b$ ●

Remark. Use the partition to think of $[a,b] = \bigcup_{i=1}^n [x_{i-1}, x_i]$.

A.2 $\inf_A f = \inf f_A$ & $\sup_A f = \sup f_A$ $\forall A \subseteq$ domain of a real-valued map f ●

A.3 $\forall$bounded map $[a,b] \xrightarrow{f} \mathbb{R}$ $\forall$partition $P = \{x_i\}_{i=0}^n$ of $[a,b]$, **Riemann's *lower & upper sums*** are

$$L_{f,P,[a,b]} = \sum_{i=1}^n (x_i - x_{i-1}) \inf_{[x_{i-1},x_i]} f \quad \& \quad U_{f,P,[a,b]} = \sum_{i=1}^n (x_i - x_{i-1}) \sup_{[x_{i-1},x_i]} f$$ ■

Remark. Riemann's sums approximate the signed area under f's graph.

A.1 *$\forall$bounded map $[a,b] \xrightarrow{f} \mathbb{R}$ $\forall$partitions P, P' of $[a,b]$ with the list defining P a subset of the list defining P', $L_{f,P,[a,b]} < L_{f,P',[a,b]} < U_{f,P',[a,b]} < U_{f,P,[a,b]}$* ■

A.2 *$\forall$bounded map $[a,b] \xrightarrow{f} \mathbb{R}$ $\forall$partitionsP, P' of $[a,b]$, $L_{f,P,[a,b]} \le U_{f,P',[a,b]}$* ■

A.4 $\forall$bounded map $[a,b] \xrightarrow{f} \mathbb{R}$, **Riemann's *lower & upper integrals*** are

$$L_{f,[a,b]} := \sup_P L_{f,P,[a,b]} \quad \& \quad U_{f,[a,b]} := \inf_P U_{f,P,[a,b]}$$ ■

A.3 *$\forall$bounded map $[a,b] \xrightarrow{f} \mathbb{R}$, $L_{f,[a,b]} \le U_{f,[a,b]}$* ■

A.5 A bounded map on a closed bounded interval is **Riemann *integrable*** if its lower and upper Riemann integrals are equal. *E.g.* **Riemann's *integral*** $\int_a^b f = L_{f,[a,b]} = U_{f,[a,b]}$ of a Riemann integrable map $[a,b] \xrightarrow{f} \mathbb{R}$ ●

Example A.1 $\forall [0,1] \xrightarrow{f:x \mapsto x^2} \mathbb{R}$ $\forall P_n = \{i/n\}_{i=0}^n$,

$$L_{f,P_n,[0,1]} = \sum_{i=1}^n \frac{1}{n}\left(\frac{i-1}{n}\right)^2 = \frac{1}{3} - \frac{1}{2n} + \frac{1}{6n^2},$$

$$U_{f,P_n,[0,1]} = \sum_{i=1}^n \frac{1}{n}\left(\frac{i}{n}\right)^2 = \frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2};$$

$$U_{f,[0,1]} \le \inf_{n\in\mathbb{Z}_{>0}} U_{f,P_n,[0,1]} = \boxed{\int_0^1 f = \frac{1}{3}} = \inf_{n\in\mathbb{Z}_{>0}} L_{f,P_n,[0,1]} \le L_{f,[0,1]}.$$

A.4 *Every continuous real-valued map on a closed bounded interval (and thus the map is uniformly continuous) is* Riemann *integrable* ■

A.5 *$\forall$*Riemann *integrable map $[a,b] \xrightarrow{f} \mathbb{R}$,*

$$(b-a)\inf_{[a,b]} f \le \int_a^b f \le (b-a)\sup_{[a,b]} f$$ ■

A.2 Riemann's integral is not good enough

Riemann's integration does not

- handle maps with many discontinuities or maps unbounded
- work well with limits

Example A.2 $f_{x\in[0,1]} = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{R}\backslash\mathbb{Q} \end{cases}$ has many discontinuities, and

$$\inf_{[a,b]} f = 0 \neq 1 = \sup_{[a,b]} f \xLeftarrow{\forall [a,b]\subseteq[0,1]} \exists r \in \left(\mathbb{R}\backslash\mathbb{Q}\right)^{\in[a,b]} \wedge \exists q \in \mathbb{Q}_{\in[a,b]}.$$

Thus $L_{f,P,[0,1]} = 0 \neq 1 = U_{f,P,[0,1]}$ $\forall$partition P of $[0,1]$, $L_{f,[0,1]} = 0 \neq 1 = U_{f,[0,1]}$, and $[0,1] \xrightarrow{f} \mathbb{R}$ not Riemann integrable.

Example A.3 $f_x = \begin{cases} 1/\sqrt{x} & \text{if } x \in (0,1] \\ 0 & \text{if } x = 0 \end{cases}$ is unbounded, and $\sup f_{[x_0,x_1]} = \infty$ $\forall$partition $P = \{x_i\}_{i=0}^n \Rightarrow U_{f,P,[0,1]} = \infty$ by definition. However, we may redefine $\int_0^1 f$ as $\lim_{a\downarrow 0} \int_a^1 f$, for the area under f's graph is $\lim_{a\downarrow 0}\left(\int_a^1 f = 2 - 2\sqrt{a}\right) = 2$.

Example A.4 Given a sequence $r_1, r_2, \ldots$ that includes each $q \in \mathbb{Q}_{\in[0,1]}$ exactly once but no other numbers, and $f_{k\in\mathbb{Z}_{>0}, x\in[0,1]} = \begin{cases} 1/\sqrt{x-r_k} & \text{if } x > r_k \\ 0 & \text{if } x \leq r_k, \end{cases}$ then $f_x = \sum_{k=1}^{\infty} f_{k;x}/2^k$ is unbounded on every non-empty open subinterval $I \subseteq [0,1]$ because $I \ni q \in \mathbb{Q}$, and f's Riemann integral is thus undefined on I, although the area (< 2) under f's graph seems reasonable.

Example A.5 Riemann's integration does not work well with pointwise limits. *E.g.* given a sequence $r_1, r_2, \ldots$ that includes each $q \in \mathbb{Q}_{\in[0,1]}$ exactly once but no other numbers, then each $f_{k\in\mathbb{Z}_{>0}, x\in[0,1]} = \begin{cases} 1 & \text{if } x \in \{r_i\}_{i=1}^k \\ 0 & \text{otherwise} \end{cases}$ is Riemann integrable and $\int_0^1 f_k = 0$. However, $f_x = \lim_{k\to\infty} f_{k;x} = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{R}\backslash\mathbb{Q} \end{cases}$ is not Riemann integrable (*cf.* example A.2).

A.6 *$\forall$sequence $f_1, f_2, \ldots$ of* Riemann *integrable maps on* $[a,b]$ *with* $|f_{k\in\mathbb{Z}_{>0}, x\in[a,b]}| \leq M \in \mathbb{R}$, $\int_a^b f = \lim_{k\to\infty} \int_a^b f_k$ *if*

1. $\forall x \in [a,b] \exists f_x = \lim_{k\to\infty} f_{k,x}$
2. f *is* Riemann *integrable on* $[a,b]$ ■

Remark. The undesirable hypothesis 2 and the difficulty in finding a simple Riemann-integration-based proof suggest that Riemann's integration is not the ideal integration theory.

B Complete ordered fields

B.1 A ***field*** is a set $\mathbb{F}$ with two binary operations symbolised as addition and multiplication: $\forall a \,\&\, b \,\&\, c \in \mathbb{F}$

Commutativity $a + b = b + a \wedge ab = ba$

Associativity $(a + b) + c = a + (b + c) \wedge (ab)c = a(bc)$

Multiplicative distributivity over addition $a(b + c) = ab + ac$

Additive identity $\exists! \mathbf{0}_{\mathbb{F}} \in \mathbb{F} : a + \mathbf{0} = a$

Multiplicative identity $\exists! \mathbf{1}_{\mathbb{F}} \in \mathbb{F} : a\mathbf{1} = a$

Additive inverse $\exists! -a \in \mathbb{F} : a + (-a) = \mathbf{0}$

Multiplicative inverse $\exists! a^{-1} \in \mathbb{F} : aa^{-1} = \mathbf{1}$ ●

Remark 2. $-(-a = -\mathbf{1} \cdot a) = a \overset{\neq \mathbf{0}}{=\!=} \left(a^{-1}\right)^{-1} \forall a \in \mathbb{F}$.

E.g. The set $\mathbb{Q}$ of rationals under usual addition and multiplication.

E.g. The set $\{0, 1\}$ under usual addition and multiplication except that $1 + 1 := 0$.

B.1 $a\mathbf{0} = \mathbf{0}\ \forall a \in$ *field* $\mathbb{F}$ ■

B.2 $\forall a, b \in$ field $\mathbb{F}$, their

Difference $a - b := a + (-b)$

Quotient $a/b := ab^{-1}$ for $b \neq \mathbf{0}$ ●

B.3 A field $\mathbb{F}$ is ***ordered*** if $\exists$ ***positive*** $P \subset \mathbb{F}$:

- $a \in \mathbb{F} \Rightarrow a \in P \veebar a = \mathbf{0} \veebar -a \in P$
- $a \,\&\, b \in P \Rightarrow a + b \in P \wedge ab \in P$ ●

B.2 *A positive* $P \subset$ *ordered field* $\mathbb{F}$ *is closed under multiplicative inverse; i.e.* $a^{-1} \in P$ $\forall a \in P$, *with* $\mathbf{1} \in P$ ■

B.4 $\forall a \,\&\, b \in$ ordered field $\mathbb{F} \supset$ positive P

- $a < b \iff b - a \in P \iff b > a$
- $a \leq b \iff a < b \vee a = b \iff b \geq a$ ●

Remark 3. $\mathbf{0} < b$ iff $b \in P$.

B.3 *The ordering* $<$ *on an ordered field* $\mathbb{F}$ *is* ***transitive****; i.e.* $a < b < c \xRightarrow{\forall a,b,c \in \mathbb{F}} a < c$ ■

B.5 The ***absolute value*** $|b| := \begin{Bmatrix} b & \text{if } b \geq \mathbf{0} \\ -b & \text{if } b < \mathbf{0} \end{Bmatrix}$ of $b \in$ ordered field $\mathbb{F}$ ●

Remark 4. $|b| \geq b, -b$.

B.4 $|a + b| \leq |a| + |b|\ \forall a \,\&\, b \in$ *ordered field* $\mathbb{F}$ ■

B.5 *Every ordered field* $\mathbb{F} \supseteq \mathbb{Q}$*; i.e.* $\exists$ *injection*[xviii] $\mathbb{Q} \xrightarrow{\varphi} \mathbb{F}$, *such that*

$$\varphi_{\pm m/n} := (\underbrace{\pm\mathbf{1} \pm \cdots \pm \mathbf{1}}_{m\ times})(\underbrace{\mathbf{1} + \cdots + \mathbf{1}}_{n\ times})^{-1} \overset{m=0}{=\!=} \mathbf{0} =: \varphi_0$$

$\forall m \in \mathbb{Z}_{\geq 0} := \{z \in \mathbb{Z} | z \geq 0\}\ \forall n \in \mathbb{Z}_{>0}$, *preserving all ordered field properties.*[xix] ■

B.6 $q^2 = 2 \Rightarrow q \notin \mathbb{Q}$ ■

B.6 $b \in$ ordered field $\mathbb{F}$ is an ***upper bound*** of $A \subseteq \mathbb{F}$ if $a \leq b \in \mathbb{F}\ \forall a \in A$ ●

E.g. For both $\mathbb{Q}_{\leq 3}$ and $\mathbb{Q}_{<3}$, every $b \in \mathbb{Q}_{\geq 3}$ is an upper bound, and 3 is the ***least*** upper bound.

Remark 5. A least upper bound of a set, if it exists, is unique.

Example B.1 $\mathbb{Q}_{<\sqrt{2}} = \{q \in \mathbb{Q} | q^2 < 2\}$ has no least upper bound $b \in \mathbb{Q}$. The idea is that

- $b \in \mathbb{Q}_{<\sqrt{2}} \Rightarrow \exists b'$ ($= \left[b + {}^{(2-b^2)}\!/_5\right]$ for example) $\in \mathbb{Q}_{<\sqrt{2}}$ slightly bigger than b
- $b \in \mathbb{Q}_{>\sqrt{2}} \Rightarrow \mathbb{Q}_{<\sqrt{2}}$ has an upper bound ($\left[b - {}^{(b^2-2)}\!/_{2b}\right]$ for example) slightly smaller than b

[xviii] *I.e.* $\varphi_{m/n} = \varphi_{p/q} \xLeftrightarrow{\forall m,n,p,q \in \mathbb{Z}_{>0}} m/n = p/q$

[xix] *Viz.*, $\forall a \,\&\, b \in \mathbb{Q}$, $\varphi_{a+b} = \varphi_a + \varphi_b$, $\varphi_{ab} = \varphi_a \varphi_b$, $\varphi_a > 0 \iff a > 0$ *etc.* (with $a \neq 0$ for the multiplicative inverse condition)

- So $b = \sqrt{2} \notin \mathbb{Q}$.

B.7 An ordered field is ***complete*** if every its non-empty subset ***bounded above*** has a least upper bound; denote the field by $\mathbb{R}$ and call it the field of ***real numbers*** ●

B.8 $\tilde{r}$ is **Dedekind's *cut*** if

- $\varnothing \subset \tilde{r} \subset \mathbb{Q}$
- $q \in \mathbb{Q}_{<r\in\tilde{r}} \Rightarrow q \in \tilde{r}$
- $\tilde{r}$ has no largest element

Denote the set of all Dedekind's cuts by $\tilde{\mathbb{R}}$ ●

Remark 6. Intuitively, $\tilde{r} = \mathbb{Q}_{<r} \approx r \in \mathbb{R} \approx \tilde{\mathbb{R}}$.

B.9 $S \backslash A := \{s \in S | s \notin A\}$ is the ***set difference*** from A to S. If $A \subseteq S$, then $S\backslash A$ is A's ***complement*** in S ●

B.10 ***Make*** $\tilde{\mathbb{R}}$ ***a field*** $\forall \tilde{r}_{i=1,2} \in \tilde{\mathbb{R}}, \tilde{\mathbb{R}} \ni$

- $\sum_{i=1,2} \tilde{r}_i := \left\{\sum_{i=1,2} r_i \middle| r_{j=1,2} \in \tilde{r}_j\right\}$
- $\tilde{0} := \mathbb{Q}_{<0}$
- $-\tilde{r} := \left\{r \in \mathbb{Q} \middle| (\mathbb{Q}\backslash\tilde{r})^{<-r} \neq \varnothing\right\}$
- $\prod_{i=1}^{2} \tilde{r}_i := \begin{cases} \left\{\prod_{i=1}^{2} r_i \middle| r_{j=1,2} \in \tilde{r}_j^+\right\} \cup \mathbb{Q}_{\leq 0} & \text{if } \tilde{r}^+_{j=1,2} \neq \varnothing \\ \left\{\prod_{i=1}^{2} r_i \middle| r_j \in \tilde{r}_j, r_{3-j} \in \mathbb{Q}\backslash\tilde{r}_{3-j}\right\} & \text{if } \tilde{r}^+_{j\in\{1,2\}} = \varnothing \neq \tilde{r}^+_{3-j} \\ \left\{q \in \mathbb{Q} \middle| \exists r_{i=1,2} \in \tilde{r}_i^- : q < \prod_{i=1}^{2} r_i\right\} & \text{if } \tilde{r}^+_{j=1,2} = \varnothing \end{cases}$ with

$\tilde{r}^+ := \tilde{r}^{>0}$[xx] and $\tilde{r}^- := (\mathbb{Q}\backslash\tilde{r})^{\leq 0}$

- $\tilde{1} := \mathbb{Q}_{<1}$
- $\tilde{r}^{-1} := \left\{r \in \mathbb{Q} \middle| (\mathbb{Q}\backslash\tilde{r})^{<r^{-1}} \neq \varnothing\right\}$

Make field $\tilde{\mathbb{R}}$ ***ordered*** define $\tilde{r} \in \tilde{\mathbb{R}}$ to be ***positive*** if $\exists b \in \tilde{r} : b > 0$[xxi] ●

B.7 *The ordered field* $\tilde{\mathbb{R}}$ *is complete; i.e.* $\varnothing \subset \tilde{R} \subset \tilde{\mathbb{R}} \wedge \tilde{R}$ *bounded above* $\Rightarrow \tilde{R}$ *has a least upper bound* $\bigcup_{\tilde{r}\in\tilde{R}} \tilde{r}$ ■

C Supremum & infimum

Property (Archimedian) $\forall r \in \mathbb{R}\, \exists z \in \mathbb{Z}_{>0} : r < z$. *I.e.* $\forall r \in \mathbb{R}^{>0}\, \exists z \in \mathbb{Z}_{>0} : z^{-1} < r$ ■

C.1 $\forall a \in \mathbb{R}^{<b\in\mathbb{R}}\, \exists q \in \mathbb{Q}_{\in(a,b)}$ ■

C.1 $b \in \mathbb{R}$ is a ***lower bound*** of $A \subseteq \mathbb{R}$ if $b \leq a\, \forall a \in A$ ●

E.g. For both $\mathbb{R}^{>3}$ and $\mathbb{R}^{\geq 3}$, every $b \in \mathbb{R}^{\leq 3}$ is a lower bound, and 3 is the ***greatest*** lower bound.

Remark 7. A greatest lower bound of $A \subseteq \mathbb{R}$, if it exists, is unique.

C.2 *Every non-empty* $A \subseteq \mathbb{R}$ ***bounded below*** *has a greatest lower bound* ■

C.2 $\forall A \subseteq \mathbb{R}$, its ***supremum*** & ***infimum*** are respectively

$$\sup_A := \begin{cases} A\text{'s least upper bound} & \text{if } A \text{ bounded above} \wedge A \neq \varnothing \\ \infty & \text{if } A \text{ has no upper bound} \\ -\infty & \text{if } A = \varnothing \end{cases}$$

[xx] Think of the condition $\tilde{r}^+ \neq \varnothing$ as equivalent to $\tilde{r} > \tilde{0}$

[xxi] $\tilde{r}_1 \subset \tilde{r}_2 \iff \tilde{r}_1 < \tilde{r}_2 \overset{\text{definition}}{\iff} (\tilde{r}_2 - \tilde{r}_1)$ positive

$$\& \ \inf_A := \begin{cases} A\text{'s greatest lower bound} & \text{if } A \text{ bounded below} \wedge A \neq \varnothing \\ -\infty & \text{if } A \text{ has no lower bound} \\ \infty & \text{if } A = \varnothing \end{cases}$$ ■

C.3 $r \in \mathbb{R}$ is ***irrational*** if $r \notin \mathbb{Q}$; *i.e.* $r \in \mathbb{R}\backslash\mathbb{Q}$ ●

C.3 $\exists r \in \mathbb{R}^{>0} : r^2 = 2$. *I.e.* $\exists r = \sqrt{2} \in \mathbb{R}\backslash\mathbb{Q}$ ●

C.4 $\forall a \in \mathbb{R}^{<b\in\mathbb{R}} \exists r \in \left(\mathbb{R}\backslash\mathbb{Q}\right)^{\in(a,b)}$ ■

C.4 $(-\infty, \infty) := \mathbb{R}$, with

- the ordering $>$ on $\mathbb{R}$ extended to $[-\infty, \infty] := \mathbb{R} \cup \{\pm\infty\}$ as
  - $a < \infty \,\forall a \in [-\infty, \infty) := \mathbb{R} \cup \{-\infty\}$
  - $-\infty < a \,\forall a \in (-\infty, \infty] := \mathbb{R} \cup \{\infty\}$
- $\forall a, b \in [-\infty, \infty]$
  - $a < b \iff b > a$
  - $a \leq b \iff a < b \vee a = b \iff b \geq a$ ●

C.5 $I \in [-\infty, \infty]$ is an ***interval*** if $(a, b) \subseteq I \,\forall a, b \in I$ ●

C.5 $\forall$*interval* $I \in [-\infty, \infty]$ $\exists a \,\&\, b \in [-\infty, \infty] : (a, b) \subseteq I \subseteq [a, b]$. *So* $I = (a, b) \vee [a, b] \vee (a, b] \vee [a, b)$ ■

D Open & closed subsets of $\mathbb{R}^n$

D.1 $\mathbb{R}^n := \left\{(x_1, \ldots, x_n) \equiv (x_i)_{i=1}^n \middle| x_{j=1,\ldots,n} \in \mathbb{R}\right\}$ is the set of all ordered n-tuples of real numbers ●

D.2 $\forall x = (x_i)_{i=1}^n \in \mathbb{R}^n$, $\|x\| := \sqrt{\sum_{i=1}^n |x_i|^2}$, $\|x\|_\infty := \max\{|x_i|\}_{i=1}^n$ ●

D.3 A sequence $a_1, a_2, \cdots \in \mathbb{R}^n$ ***converges*** to a ***limit*** $L = \lim_{k\to\infty} a_k$ if $\forall \epsilon > 0$ $\exists m \in \mathbb{Z}_{>0} : \left\|a_{\forall k\geq m} - L\right\|_\infty < \epsilon$ ●

Remark 8. $\lim_{k\to\infty} a_k = L \overset{\text{definition D.3}}{\iff} \lim_{k\to\infty} \left\|a_k - L\right\|_\infty = 0$

$\overset{\|x\|_\infty \leq \|x\| \leq \sqrt{n}\|x\|_\infty \,\forall x \in \mathbb{R}^n}{=\!=\!=} \lim_{k\to\infty} \left\|a_k - L\right\|$.

D.1 *A convergent sequence* $a_1, a_2, \cdots \in \mathbb{R}^n$ *converges coordinate-wise; i.e.* $\lim_{k\to\infty}\left(a_k = (a_{k,j})_{j=1}^n\right) = L = (L_j)_{j=1}^n$ *iff* $\lim_{k\to\infty} a_{k,\forall j\in\{i\}_{i=1}^n} = L_j$ ■

D.4 $\forall x \in \mathbb{R}^n \,\forall \delta > 0$, the ***open cube*** $B_{x,\delta} := \left\{y \in \mathbb{R}^n \middle| \|y - x\|_\infty < \delta\right\}$ ●

D.5 An ***open interval*** $I = (a, b) \subseteq \mathbb{R}$ for some $a, b \in [-\infty, \infty]$ ●

D.6 $X \subseteq \mathbb{R}^n$ is

Open if $B_{\forall x\in X, \exists \delta>0} \subseteq X$

Closed if its complement in $\mathbb{R}^n$ is open ●

Remark 9. Instead of open cubes, open sets could have been equivalently defined using open balls $\{y \in \mathbb{R}^n | \|y - x\| < \delta\} \subseteq B_{x,\delta} \subseteq \left\{y \in \mathbb{R}^n \middle| \|y - x\| < \sqrt{n}\delta\right\}$.

D.7 $\forall$collection $\mathcal{A}$ of a set S's subsets, the ***union*** $\bigcup_{E\in\mathcal{A}} E := \{x \in S | \exists E \in \mathcal{A} : x \in E\}$ and the ***intersection*** $\bigcap_{E\in\mathcal{A}} E := \{x \in S | x \in E \,\forall E \in \mathcal{A}\}$ ■

E.g. $\bigcup_{k=1}^\infty [1/k, 1 - 1/k] = (0, 1)$, $\bigcap_{k=1}^\infty (-1/k, 1/k) = \{0\}$.

D.2 *The union of every collection of open subsets of* $\mathbb{R}^n$ *is open in* $\mathbb{R}^n$*; so as the intersection of every finite collection of open subsets of* $\mathbb{R}^n$ ■

D.8 A set C is ***countable*** if $C = \varnothing \vee C = \{c_1, c_2, \ldots\}$ for some sequence $c_1, c_2, \ldots$ of elements of C ●

Remark. Every finite set is countable. If C is infinite countable, then it can be written as $\{b_1, b_2, \dots\}$ of distinct elements.

D.3 *$\mathbb{Q}$ is countable* ■

Proof. Start with the list $\{-1, 0, 1\}$ at step 1, adjoin to the list in increasing order the rationals $\in [-n, , n]$ that can be written in the form m/n for some $m \in \mathbb{Z}$ at step n, and continue in this fashion to produce a sequence containing each rational □

D.9 A sequence $E_1, E_2, \dots$ of sets is ***disjoint*** if $E_{\forall j \neq k} \cap E_k = \varnothing$ ●

D.4 *$A \subseteq \mathbb{R}$ open iff A the countable disjoint union of open intervals* ■

D.5 *$A \subseteq \mathbb{R}^n$ closed iff $A \ni$ limit of every convergent sequence of elements of A* ■

Laws (de Morgan's) *$\forall$collection $\mathcal{A}$ of subsets of some set X,* $X \setminus \bigcup_{E \in \mathcal{A}} E = \bigcap_{E \in \mathcal{A}} (X \setminus E)$, $X \setminus \bigcap_{E \in \mathcal{A}} E = \bigcup_{E \in \mathcal{A}} (X \setminus E)$ ■

D.6 *The intersection of every collection of closed subsets of $\mathbb{R}^n$ is closed in $\mathbb{R}^n$; so as the union of every finite collection of closed subsets of $\mathbb{R}^n$* ■

D.7 *The only subsets of $\mathbb{R}^n$ that are both open and closed are $\varnothing$ and $\mathbb{R}^n$* ■

E Sequences & continuity

E.1 A sequence $a_1, a_2, \dots \in \mathbb{R}$ is

Increasing if $a_{\forall k \in \mathbb{Z}_{>0}} \leq a_{k+1}$

Decreasing if $a_{\forall k \in \mathbb{Z}_{>0}} \geq a_{k+1}$

Monotone if it is either increasing or decreasing ●

E.2 • $A \subseteq \mathbb{R}^n$ is ***bounded*** if $\sup \{\|a\|_\infty\}_{a \in A} < \infty$

• A map into $\mathbb{R}^n$ is ***bounded*** if its range is a bounded subset of $\mathbb{R}^n$. Particularly, a sequence $a_1, a_2, \dots \in \mathbb{R}^n$ is bounded if $\sup \left\{\left\|a_k\right\|_\infty\right\}_{k \in \mathbb{Z}_{>0}} < \infty$ ●

E.1 *Every bounded monotone sequence of real numbers converges* ■

E.3 $a_{k_1}, a_{k_2}, \dots$, with $k_{i=1,2,\dots} \in \mathbb{Z}_{>0}$ and $k_1 < k_2 < \cdots$, is a ***subsequence*** of a sequence $a_1, a_2, \dots$ ●

E.2 *Every sequence of real numbers has a monotone subsequence* ■

E.3 (**Bolzano-Weierstrass's**) *Every bounded sequence in $\mathbb{R}^n$ has a convergent subsequence* ■

E.4 *Every sequence of elements of a closed bounded $F \subseteq \mathbb{R}^n$ has a subsequence that converges to an element of F* ■

E.4 $A \xrightarrow{f} \mathbb{R}^n \; \forall A \subseteq \mathbb{R}^m$ is ***continuous***

At $b \in A$ if $\forall \epsilon > 0 \, \forall a \in A \, \exists \delta > 0 : \|a - b\|_\infty < \delta \Rightarrow \|f_a - f_b\|_\infty < \epsilon$

On A if it is continuous at every $b \in A$ ●

E.5 *$A \xrightarrow{f} \mathbb{R}^n \; \forall A \subseteq \mathbb{R}^m$ is continuous at $b \in A$ iff $f_{b_k} \xrightarrow{k \to \infty} f_b$ $\forall$sequence $b_{k=1,2,\dots} \in A$ that converges at b* ■

E.5 $A \xrightarrow{f} \mathbb{R}^n \quad \forall A \subseteq \mathbb{R}^m$ is ***uniformly continuous*** if $\forall \epsilon > 0 \quad \exists \delta > 0 \quad \forall a, b \in A : \|a - b\|_\infty < \delta \Rightarrow \|f_a - f_b\|_\infty < \epsilon$ ●

Example E.1 $\mathbb{R} \xrightarrow{f: x \mapsto x^2} \mathbb{R}$ is continuous but not uniformly continuous.

E.6 *Every continuous $\mathbb{R}^n$-valued map on a closed bounded subset of $\mathbb{R}^m$ is uniformly continuous* ■

E.7 *Every continuous real-valued map of a closed bounded subset of* $\mathbb{R}^m$ *attains its maximum and minimum* ■

E.6 $\forall S \xrightarrow{f} T$ between sets S and T, $f_X := \{f_x\}_{x \in X}$ is the ***image*** of $X \subseteq S$ under f ●

E.8 *A continuous* $F \xrightarrow{f} \mathbb{R}^n$ *of a closed bounded* $F \subseteq \mathbb{R}^m$ *is a closed bounded subset of* $\mathbb{R}^n$

■